AN IMPROVED PASSIVITY PRESERVING MODEL ORDER REDUCTION TECHNIQUE

By

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A thesis submitted to the faculty of Information Security Department, Military College of Signals, National University of Sciences and Technology, Islamabad, Pakistan, in partial fulfilment of the requirements for the degree of MS in Systems Engineering

April 2019

ABSTRACT

The design of any system e.g. network systems and telecommunication systems contain mathematical model which are complex and require lot of computational power for analysis and simulation. For such scenarios, to make the analysis and design of these systems easier, model order reduction (MOR) is used which reduce the complexity of these models by reducing their order. It takes less computational power and simulation time when models with reduced orders are used. MOR methods are very useful when it comes to analyze large and complex systems such as space systems, state estimation in UAVs and high voltage systems etc.

In feedback control systems theory, the major contribution of balanced truncation is its application in model reduction which gives the stable reduced models and an error bound within certain frequency limit. This is also known as frequency weighted MOR problem. For a given transfer function there are almost infinite state space realizations but a particular realization has been proved useful in control systems theory which is called internally balanced realization. It indicates the dominant system states and it is the minimal realization which is also asymptotically stable.

The most wanted property in some systems is passivity which can be implied if the system's transfer function is positive real. When the model is reduced to r^{th} ROM using MOR it is desired to preserve the important features such as passivity, stability and input output behavior etc. Since passive systems are also stable systems and not vice versa, it is very important in MOR algorithms to preserve passivity.

A lot of work has been done on passivity preserving model order reduction techniques in case of continuous time (CT) systems whereas no work is done in case of discrete time (DT) systems. Performance of a system can be enhanced by sampling and also the computational cost can be relatively reduced in this case.

This research focuses on passivity preserving model order reduction (MOR) technique for discrete time systems. Balanced truncation along with extended Enns' and Umair et al. technique proposed for continuous time systems are modified for discrete time systems. The proposed technique preserves passivity and yields reasonable approximation error.

DEDICATION

This thesis is dedicated to

MY BELOVED PARENTS, SIBLINGS,

HONORABLE TEACHERS AND FRIENDS

for their love, endless support and encouragement

ACKNOWLEDGEMENTS

I am grateful to God Almighty who has bestowed me the strength and the passion to accomplish this thesis and I am thankful to Him for His mercy and benevolence, without whose consent I could not have indulged myself in this task.

The achievement of this thesis is accomplished with the participation and co-operation of all guidance committee. With affection and deep appreciation, I acknowledge my indebtedness to his honorable thesis supervisor Dr. Muhammad Imran, who not only supported me with every possible guidance but also encouraged my spirits at critical junctures that lead to the successful completion of this thesis.

I express my sincere gratitude to Col. Dr. Abdul Ghafoor, Associate Professor, Military College of Signals (NUST), from whom I sought valuable advice and encouragement to achieve this task.

The loving support from my family and friends has been invaluable, whose courage and helps enable me to complete my Master Degree in Systems Engineering and especially in thesis.

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Chapter 1

Introduction

1.1 Overview of Model order reduction

Mathematical modeling is critical in design, simulation, analysis and development of systems. Mathematical models (representing original systems) are often very complex which motivates the use of model order reduction (MOR) schemes which reduces complexity of systems by yielding reduced order model (ROMs) [1,2]. Good MOR schemes preserve original system's key properties like stability, passivity and input-output behavior in ROM with less approximation error [3,4,5,6,7]. Approximation error is one of important factors in MOR which is computed from the difference between output of original system and ROM for a given input. Certain numerical properties such as accuracy, computational speed play important role in determining efficiency of MOR techniques.

1.2 Summary of Model order reduction techniques

Some of the stability and passivity preserving techniques are summarized in this section in case of frequency weightings.

1.2.1 Frequency Weighted Model Reduction

Balanced truncation (BT) was proposed in [1] to compute low-order approximation of a given model by neglecting states that moderately affects the overall model response. Error bound formula is also available for BT.

In ideal case the approximation error should be minimal for all frequencies but often approximation in certain frequency range is more important than others e.g controller reduction [8]. This motivates use of frequency weights in MOR and is reffered to as frequency weighted MOR (FWMOR) [9].

FWMOR was first proposed by Enns' by extending balanced truncation [1] to include frequency weightings. Enns' method ensures stability only in case of single sided weightings [28]. Lin and Chiu presented modified Enns' technique which guarantees stability in case of double sided frequency weightings [10]. However, this technique can't be used in controller reduction application because of pole zero cancellation. In [11] Enns' technique was modified which guarantees stability in case of double sided weightings. Enns' technique was further modified in [12, 13] for continuous and discrete time systems respectively, to ensure stability in case of double sides weightings along with error bounds.

1.2.2 Passivity Preserving Model order reduction

Passivity is widely adapted tool which is used for the analysis of stability of interconnection of dynamical systems [14] and also it is used as analysis tool for mechanical and electrical systems and other domains of engineering. In particular passivity plays important role in robotics as it provides key concept for stability analysis of human-machine interaction [15], to study about the robustness of force feedback controllers, [16, 17] and to analyze the stability of tele-manipulation [18]. Passivity is often considered as a primary design constraint in the development of robotic manipulator controllers [19, 20, 21, 22]. The aim of MOR is to preserve fundamental properties of original system like input output behaviour, stability and passivity in ROM with less approximation error [23]. Since passive systems are also stable systems and not vice versa [24], it is of vital importance in MOR algorithms to preserve passivity. Another extension of balanced truncation [1] was introduced by Phillips et al. [25]. This method preserves passivity along with original system's stability for continuous time systems.

In [26] spectrally weighted BT technique is explained which claims that ROM is guaranteed to be passive. Error bounds equations are derived which are also spectrally weighted. For large scale RLC systems passivity preserving FWMOR are also discussed in [27]. Stability preserving techniques discussed in [28], [10] and [9] are also modified for passive reduced models under certain derived conditions in [27]. It has been proved in [29] that spectrally weighted BTR used in [26] fails to produce stable let alone passive ROM when it contains both input and output weights and also that this technique only works for single sided weighted case. In [24] a FW passivity preserving technique is proposed for balanced MOR. This technique produces passive ROM in case of double sided weights. In [30], a new passivity preserving frequency weighted balanced MOR technique is proposed. It also preserves passivity in case of double sided weights.

1.3 Problem Summary

To the best of author's knowledge, there is no work done on passivity preserving frequency weighted model reduction of DT systems although work on linear time varying macro-models for unweighted case appear in [31]. In this work passivity preserving MOR techniques are proposed for unweighted and frequency weightings case.

1.4 Objectives of Research

Following are the main objectives of this research: spacing

* To develop such techniques that ensures passivity of ROM for DT systems.
* To develop an expression for error bound.

1.5 Outline of Thesis

This thesis is divided into following five chapters:

* Chapter 1: This chapter discusses the summary of existing FWMOR techniques. Chapter 2: This chapter discusses existing FWMOR techniques in detail.
* Chapter 3: This chapter deals with proposed techniques for generating guaranteed passive ROM.
* Chapter 4: Several numerical examples and simulations are presented in this chapter to support the proposed methodology.
* Chapter 5: This chapters explains about future work that can be done to better the proposed techniques and conclusion.

Chapter 2

Frequency Weighted Model Reduction: A Review

2.1 Introduction

The main focus of control engineering is to implement dynamic systems derived by the mathematical modeling of physical systems. There exist huge number of physical system in world. While performing mathematical modeling, partial differential equations (PDEs) and ordinary differential equations (ODEs) are often used to model the dynamic behavior of physical system. In case of complex systems we may get high order mathematical model of dynamic system. Different MOR techniques available in literature, are used to reduce the complexity of these model to make the analysis and design easier (see Fig 2.1). Because of the fast development of digital computers

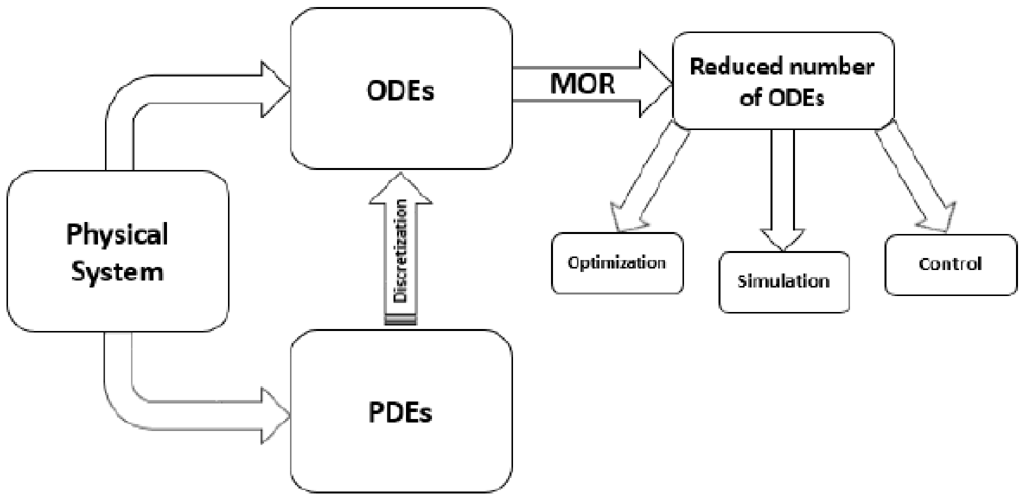

Figure 2.1: Significance of model order reduction

and their usage in control system, the importance of reduced order modeling methods also increased for DT systems. Digital control systems are often preferred because of compact size, flexibility and less susceptibility to noise. Many airborne systems contain digital controllers that holds thousands of discrete elements and takes space no larger than a regular book. Discrete time (DT) control is often desirable because better performance can be achieved by sampling. The computational cost is comparatively less than that of continuous control systems [31]. Discrete time passivity has numerous practical applications e.g design and control of haptic interface [15, 32], adaptive windowing for velocity estimation [33], analysis of haptic interaction stability with deformable objects [34]. Therefore passivity in discrete time systems is as important as it is in continuous time (CT) systems. In this chapter we will first review BT method for both CT and DT systems. In this method the fequency response of ROM follows that of original system over an infinite frequency range. After that, frequency weighted passivity preserving MOR techniques will be discussed for CT systems only, since there is no work done on DT systems.

2.2 Preliminaries

Since the techniques proposed in this thesis are based on balanced truncation [1] so it will be discussed first.

2.2.1 Balanced Truncation

Let $G(s)$ be the n^{th} order original stable system given by $G(s) = D + C(sI - A)^{-1}B$, where $\{A, B, C, D\}$ is state space realization of $G(s)$, and $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, $C \in \Re^{p \times n}$ and $D \in \Re^{p \times m}$. The controllability and observability Gramians, P_d and Q_d respectively, can be computed using following CT Lyapunov equations.

$$AP_d + P_dA^T + BB^T = 0 \tag{2.1a}$$

$$A^TQ_d + Q_dA + C^TC = 0 \tag{2.1b}$$

Both P_d and Q_d are symmetric and positive definite matrices i.e $P_d, Q_d > 0$. Let T be the contra-gradient transformation computed by diagonalizing Gramian P_d and Q_d simultaneously such that:

$$T^{-1}P_dT^T = T^TQ_dT = \Sigma_d = diag\{\sigma_1, \sigma_2, ..., \sigma_n\} \tag{2.2}$$

where σ_i are Hankel Singular Values (HSV's) and $\sigma_i \geq \sigma_{i+1}$ for $i = 1, 2, ..., n-1$. The balanced realization A_b, B_b, C_b, D_b is computed as follows:

$$
\begin{bmatrix} A_b & B_b \\ \hline C_b & D_b \end{bmatrix} = \begin{bmatrix} TAT^{-1} & T^{-1}B \\ \hline CT & D \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{bmatrix} \tag{2.3a}
$$

$$
G_r(s) = C_1(sI - A_{11})^{-1}B_1 + D_1 \tag{2.3b}
$$

The ROM $\{A_{11}.B_1, C_1, D\}$ is obtained by truncating the low energy states in (2.3) where $A_{11} \in \Re^{r\times r}$, $B_1 \in \Re^{r\times m}$, $C \in \Re^{p\times r}$ and $D \in \Re^{p\times m}$, and r is the order of the ROM $(r < n)$.

2.2.2 Properties of Balanced truncation

The properties of discrete time Balanced truncation is given beolw:

1. If the realization $\{A, B, C, D\}$ is asymptotically stable and minimal only then it can be transformed to balanced realization $\{A_b, B_b, C_b, D_b\}$.

2. A subsystem $\{A_{ii}, B_i, C_i, D\}$ where, $i = 1, 2$, which is obtained from the original system $\{A, B, C, D\}$ is stable as well as internally balanced if no diagonal entries between two subsystems are common, i.e $\sigma_k \neq \sigma_l$ where, $k = 1, ..., r,$. (where r is the order of the ROM) and $l = r+1, ..., n$, (where n is the order of the original system).

3. The error bound is expressed below:

$$
||G(s) - G_r(s)||_\infty \leq 2\Sigma_{i=r+1}^{n}\sigma_i
$$

where σ_i are the HSV's.

2.3 Motivation and Problem Formulation

Since passive systems are also stable systems whereas stability doesn't imply passivity [24], it is of vital importance in MOR algorithms to preserve passivity. Discrete time (DT) control is often desirable because better performance can be achieved by sampling. The computational cost is comparatively less than that of continuous control systems [31]. Discrete time passivity has numerous practical applications e.g design and control of haptic interface [15, 32], adaptive windowing for velocity estimation [33], analysis of haptic interaction stability with deformable objects [34]. Therefore passivity in discrete time systems is as important as it is in continuous time (CT) systems. Proposed techniques discussed in next chapter consider both stable as well as passive systems. So in this chapter, the problem of passivity preserving in MOR will be discussed, and then extension of the frequency weighting case will be elaborated.

2.3.1 Importance and Properties of Passivity Preserving Model Order Reduction

Passivity of complex transfer function $G(s)$ is implied by its positive-realness for many electrical systems of interest. $G(s)$ is called positive real (PR) if it is unable to generate energy internally, like an RLC circuit. A passive system always lies entirely in the right half of the complex plan called Nyquist plot, while non passive system lies in the left half of the complex plan (see Figure 2.2). For positive-realness, $G(s)$ must always

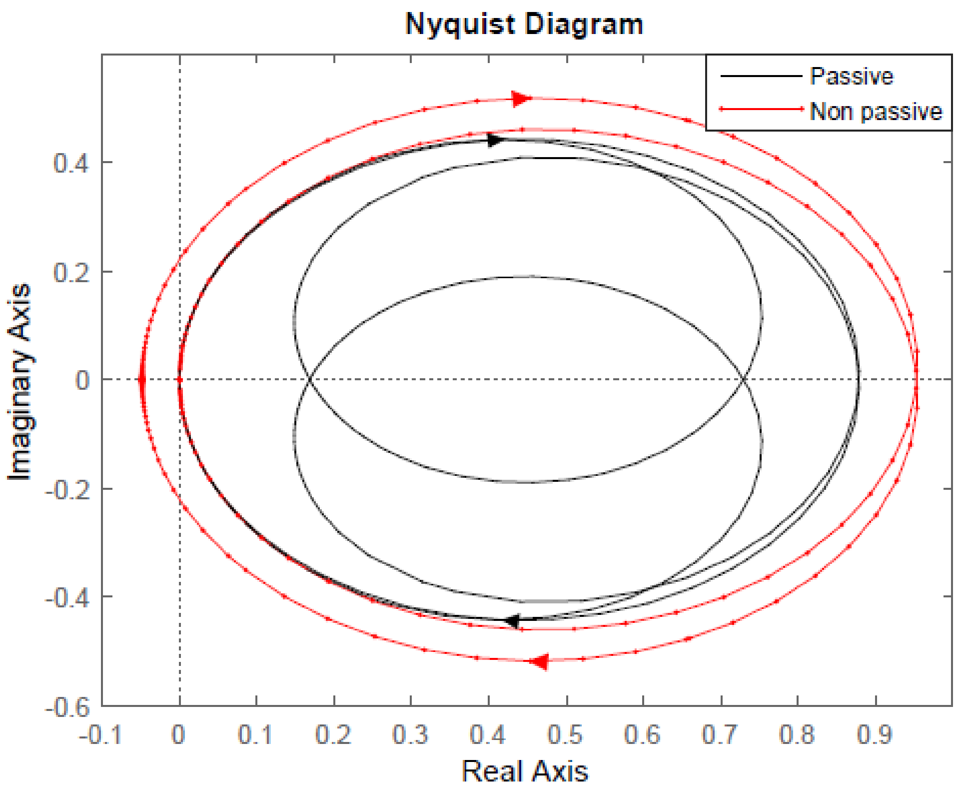

Figure 2.2: Passive and non passive systems- Nyquist Diagram

satisfy the following PR condition

$$\Re|G(s)| > 0 \; if \; \Re(s) > 0$$

where $s = \sigma + j\omega$. So the PR condition can be written as

$$\Re|G(s)| > 0 \; if \; \sigma > 0$$

and there is no constraint on $\Im(s)$. Consider an n^{th} order PR system.

$$G(s) = D + C(sI - A)^{-1}B \tag{2.4}$$

where the state space realization $\{A, B, C, D\}$ is a minimal realization of $G(s)$. Because of the unique stability properties of PR systems, they are of special interest in the analysis and design of control system. Passivity is considered to be one of the very important properties of an RLC system. Since passive system is always stable but vice versa is not true [24]. So it is necessary for a ROM $G_r(s) = D + C_r(sI - Ar)^{-1}B_r$ to preserve passivity like an original system. A passive system always satisfies the following set of Lur'e equations

$$AP_{RE} + P_{RE}A^T = -K_iK_i^T \tag{2.5a}$$

$$P_{RE}C^T - B = -K_iW_i^T \tag{2.5b}$$

$$W_iW_i^T = D + D^T \tag{2.5c}$$

$$A^TQ_{RE} + Q_{RE}A = -K_o^TK_o \tag{2.6a}$$

$$Q_{RE}B - C^T = -K_o^TW_o \tag{2.6b}$$

$$W_o^TW_o = D + D^T \tag{2.6c}$$

where, $P_{RE} > 0$ is the controllability Gramian and $Q_{RE} > 0$ is the observability Gramian of the passive system, respectively. The above Lur'e equations can be solved for P_{RE} and Q_{RE} by using the following algebraic Riccati equations (AREs) [35].

$$AP_{RE} + P_{RE}A^T + (P_{RE}C^T - B)(D + D^T)^{-1}(CP_{RE} - B^T) = 0 \tag{2.7}$$

$$A^TQ_{RE} + Q_{RE}A + (Q_{RE}B - C^T)(D + D^T)^{-1}(B^TQ_{RE} - C) = 0 \tag{2.8}$$

2.3.2 A Review of Passivity Preserving Model order Reduction

Phillips et al. [25] presented a family of algorithms for passive ROM's which are similar to the well-known BT method [1] for stable ROM. In [25] controllability and observability Gramians are obtained from Lur'e equations without using frequency weights. Muda et al. [27] extended the methods of [9, 28, 10] for RLC systems to ensure passivity, since these only yield stable ROMs. Conditions for guaranteed passivity are also given in [27] for the three extended techniques. Heydari and Pedram claimed in [26] that their technique produces guaranteed passive ROM for the double sided frequency weighting case, and the spectrally-weighted error bound is also available. The technique of [26] produces passive ROM for the single sided weighting case similar to [27] which produces stable ROM for the case when only one sided weighting is used. In [29] it has been proved that the technique of [26] may yields non-passive ROM for the passive original system in case of double-sided frequency weighting. It is also proved in [24] that the method of [26] can preserves passivity only when one sided weighting is present.

2.4 Frequency Weighted Model Order Reduction

Let an n^{th} order stable original system $G(s)$ has state space realization as $\{A, B, C, D\}$. Let v^{th} order stable input weight $G_i(s)$ and w^{th} order stable output weight $G_o(s)$ have corresponding state space realizations as $\{A_i, B_i, C_i, D_i\}$ and $\{A_o, B_o, C_o, D_o\}$ respectively, where $A_i \in \Re^{v\times v}$,$B_i \in \Re^{v\times m}$,$C_i \in \Re^{p\times v}$ and $D_i \in \Re^{p\times m}$, and $A_o \in \Re^{w\times w}$,$B_o \in \Re^{w\times m}$,$C_o \in \Re^{p\times w}$, and $D_o \in \Re^{p\times m}$. v and w represent the number of states of input and output frequency weights respectively. The objective

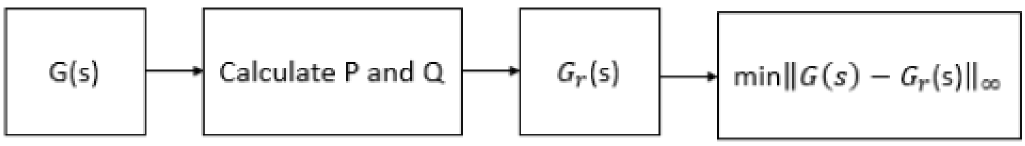

Figure 2.3: Unweighted MOR problem

of MOR is to find an r^{th} order stable ROM having state space minimal realization $\{A_r, B_r, C_r, D_r\}$, such that the error between $G(s)$ and $G_r(s)$ is made as small as possible, i.e $min||G_o(s)(G(s) - G_r(s))G_i(s)||_\infty$. This problem is so called the double sided frequency weighted MOR problem (see Figure 2.4). The problem is called single

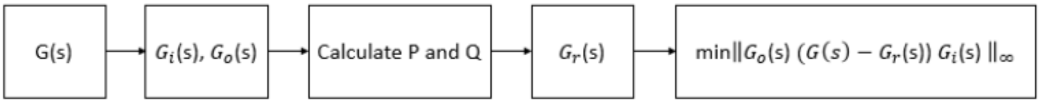

Figure 2.4: Double sided frequency weighted MOR problem

sided frequency weighted MOR problem if one of the either weights is identity. In this scenario, the objective is to minimize the error between $G(s)$ and $G_r(s)$, i.e in case of only input weight, $min||(G(s) - G_r(s))G_i(s)||_\infty$ (see Figure 2.5), and in case of only output weight, $min||G_o(s)(G(s) - G_r(s))||_\infty$ (see Figure 2.6).

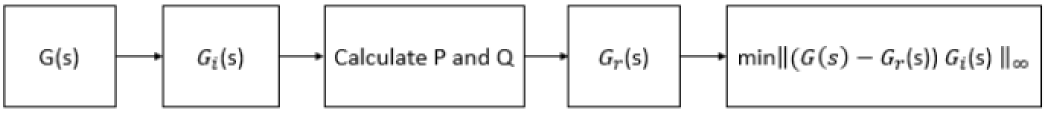

Figure 2.5: Input frequency weighted MOR problem

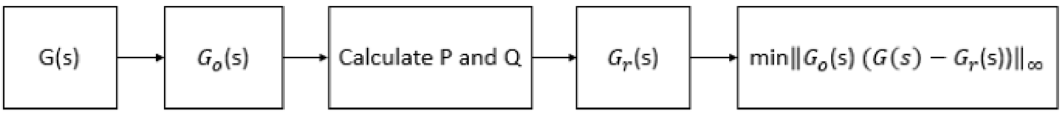

Figure 2.6: Output frequency weighted MOR problem

2.4.1 The technique of Enns'

Enns' [9] was the first to introduced frequency weights for MOR. This technique gives stable ROM for single sided frequency weighting case only. When double sided weights are used, this technique may produce unstable ROM [23]. Consider a stable original system $G(s)$ as given in (2.4). Let $G_i(s)$ and $G_o(s)$ be the input and output frequency weights

$$G_i(s) = D_i + C_i(sI - A_i)^{-1}B_i \tag{2.9a}$$

$$G_o(s) = D_o + C_o(sI - A_o)^{-1}B_o \tag{2.9b}$$

where, $\{A_i, B_i, C_i, D_i\}$ is the state space realization of input frequency weight and $\{A_o, B_o, C_o, D_o\}$ is the state space realization of output frequency weight. The input augmented system $G(s)G_i(s)$ and the output augmented system $G_o(s)G(s)$ are given as follows:

$$G(s)G_i(s) = \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline \bar{C}_i & \bar{D}_i \end{array}\right] = \left[\begin{array}{cc|c} A & BC_i & BD_i \\ 0 & A_i & B_i \\ \hline C & DC_i & DD_i \end{array}\right] \quad (2.10a)$$

$$G_o(s)G(s) = \left[\begin{array}{c|c} \bar{A}_o & \bar{B}_o \\ \hline \bar{C}_o & \bar{D}_o \end{array}\right] = \left[\begin{array}{cc|c} A & 0 & B \\ B_oC & A_o & B_oD \\ \hline D_oC & C_o & D_oD \end{array}\right] \quad (2.10b)$$

where, $\{\bar{A}_i, \bar{B}_i, \bar{C}_i, \bar{D}_i\}$ is the state space realization of input augmented system and $\{\bar{A}_o, \bar{B}_o, \bar{C}_o, \bar{D}_o\}$ is the state space realizations of output augmented system. Let $\bar{P}_E$ and $\bar{Q}_E$ satisfy the following Lyapunov equations.

$$\bar{A}_i\bar{P}_E + \bar{P}_E\bar{A}_i^T + \bar{B}_i\bar{B}_i^T = 0 \quad (2.11a)$$

$$\bar{A}_o^T\bar{Q}_E + \bar{Q}_E\bar{A}_o + \bar{C}_o^T\bar{C}_o = 0 \quad (2.11b)$$

where

$$\bar{P}_E = \begin{bmatrix} P_v & P_{12} \\ P_{12}^T & P_i \end{bmatrix} \quad (2.12a)$$

$$\bar{Q}_E = \begin{bmatrix} Q_o & Q_{12} \\ Q_{12}^T & Q_w \end{bmatrix} \quad (2.12b)$$

Remark 1 *$\bar{P}_E$ and $\bar{Q}_E$ obtained from above Lyapunov equations are symmetric and also $\bar{P}_E > 0$ and $\bar{Q}_E > 0$*

Expanding the (1,1) block of (2.11a) and (2,2) block of (2.11b) we obtain:

$$AP_b + P_vA^T + X = 0 \quad (2.13a)$$

$$A^TQ_w + Q_wA + Y = 0 \quad (2.13b)$$

where

$$X = BC_iP_{12}^T + P_{12}C_iB^T + BD_iD_i^TB^T \tag{2.14a}$$

$$Y = C^TB_o^TQ_{12}^T + Q_{12}B_oC + C^TD_o^TD_oC \tag{2.14b}$$

The matrices X and Y are generally indefinite [28], and this is the main reason of instability of a ROM in case of double sided frequency weighting. Balancing transformation matrix T which is used to diagonalize the weighted Gramians P_v and Q_w such that $T^{-1}P_vT^{-T} = T^TQ_wT = diag\{\sigma_1, \sigma_2, ..., \sigma_n\}$where σ_n are the HSV's and $\sigma_i \geq \sigma_{i+1}$ for $i = 1, 2, ..., n-1$. Then the ROM $\{A_r, B_r, C_r\} = \{A_{11}, B_1, C_1\}$ can be obtained as follows.

$$\left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right] = \left[\begin{array}{c|c} TAT^{-1} & T^{-1}B \\ \hline CT & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \tag{2.15}$$

Remark 2 *The ROM obtained by (2.15) in the presence of double sided frequency weightings may not be stable but for single sided frequency weighting it will be guaranteed to be stable [28].*

Wang et al. [28] , Varga and Anderson [8], Ghafoor and Sreeram [23], and Imran et al. [13] modified Enns' technique [9] to tackle the stability issue in case of double sided frequency weights.

2.4.2 The technique of Lin and Chiu

Lin and Chiu [10] modifies Enns' technique [9] to tackle the stability issue for the case when double sided frequency weights are used. Their technique first defines $\bar{X} =$

$P_{12}P_i^{-1}$ and $\bar{Y} = Q_o^{-1}Q_{12}^T$. Let

$$\bar{T}_i = \begin{bmatrix} I & \bar{X} \\ 0 & I \end{bmatrix} \tag{2.16a}$$

$$\bar{T}_o = \begin{bmatrix} I & -\bar{Y} \\ 0 & I \end{bmatrix} \tag{2.16b}$$

(2.16c)

be the transformations applied to the input and output augmented realizations $\{\bar{A}_i, \bar{B}_i, \bar{C}_i, \bar{D}_i\}$ and $\{\bar{A}_o, \bar{B}_o, \bar{C}_o, \bar{D}_o\}$ respectively.

$$\left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline \bar{C}_i & \bar{D}_i \end{array}\right] = \left[\begin{array}{c|c} \bar{T}_i^{-1}\bar{A}_i\bar{T}_i & \bar{T}_i^{-1}\bar{B}_i \\ \hline \bar{C}_i\bar{T}_i & \bar{D}_i \end{array}\right] = \left[\begin{array}{cc|c} A & A_{i12} & B_{i11} \\ 0 & A_i & B_i \\ \hline C & C_{i12} & DD_i \end{array}\right] \tag{2.17}$$

and

$$\left[\begin{array}{c|c} \bar{A}_o & \bar{B}_o \\ \hline \bar{C}_o & \bar{D}_o \end{array}\right] = \left[\begin{array}{c|c} \bar{T}_o^{-1}\bar{A}_o\bar{T}_i & \bar{T}_o^{-1}\bar{B}_o \\ \hline \bar{C}_o\bar{T}_o & \bar{D}_o \end{array}\right] = \left[\begin{array}{cc|c} A & A_{o12} & B \\ 0 & A_o & B_{o21} \\ \hline C_{o11} & C_o & D_oD \end{array}\right] \tag{2.18}$$

where

$$A_{i12} = AP_{12}P_i^{-1} + BC_i - P_{12}P_i^{-1}A_i \tag{2.19a}$$

$$B_{i11} = BD_i - P_{12}P_i^{-1}B_i \tag{2.19b}$$

$$C_{i12} = CP_{12}P_i + DC_i \tag{2.19c}$$

$$A_{o12} = Q_o^{-1}Q_{12}^TA + B_oC - A_oQ_o^{-1}Q_{12}^T \tag{2.19d}$$

$$B_{o12} = B_oD + Q_o^{-1}Q_{12}^TB \tag{2.19e}$$

$$C_{o11} = D_oC - C_oQ_o^{-1}Q_{12}^T \tag{2.19f}$$

Let the following Gramians be defined as:

$$\bar{P}_{LC} = \bar{T}_i^{-1} P \bar{T}_i^{-T} = \begin{bmatrix} P_n & 0 \\ 0 & P_i \end{bmatrix} \quad (2.20a)$$

$$\bar{Q}_{LC} = \bar{T}_o^T Q \bar{T}_o = \begin{bmatrix} Q_o & 0 \\ 0 & Q_n \end{bmatrix} \quad (2.20b)$$

where $P_n = P_v - P_{12} P_i^{-1} P_{12}^T$ and $Q_n = Q_w - Q_{12} Q_o^{-1} Q_{12}^T$. Let $\bar{P}_{LC}$ and $\bar{Q}_{LC}$ be the solutions of the following Lyapunov equations

$$\bar{A}_i \bar{P}_{LC} + \bar{P}_{LC} \bar{A}_i^T + \bar{B}_i \bar{B}_i^T = 0 \quad (2.21a)$$

$$\bar{A}_o^T \bar{Q}_{LC} + \bar{Q}_{LC} \bar{A}_o + \bar{C}_o^T \bar{C}_o = 0 \quad (2.21b)$$

Expanding the (1,1) and (2,2) block of (2.21a) and (2.1b) respectively, we obtain

$$A P_n + P_n A^T + B_{i11} B_{i11}^T = 0 \quad (2.22a)$$

$$A^T Q_n + Q_n A + C_{o11}^T C_{o11} = 0 \quad (2.22b)$$

Simultaneously diagonalizing the weighted controllability and observability Gramians P_n and Q_n respectively,

$$T_{LC}^{-1} P_n T_{LC}^{-T} = T_{LC}^T Q_n T_{LC} = diag\{\sigma_1, \sigma_2, ..., \sigma_n\}$$

where σ_n are the HSV's and $\sigma_i \geq \sigma_{i+1}$m for $i = 1, 2, ..., n-1$.

Remark 3 *The ROM* $\{A_r, B_r, C_r\} = \{A_{11}, B_{11}, C_1\}$ *is obtained in a same way as obtained in Enns' method [9] by truncating the low energy states in (2.15).*

Remark 4 *The realization* $\{A, B_{i11}, C_{o11}\}$ *is minimal and the ROM* $\{A_r, B_r, C_r\} = \{A_{11}, B_1, C_1\}$ *is stable. [10].*

2.4.3 The Technique of Wang et al.

Wang et al. [28] tackle the stability issue of [9] by making indefinite matrices X and Y in (2.14a) and (2.14b) positive semi-definite using eigenvalue decomposition. As we know that the matrices X and Y are symmetric matrices so we can also write X and Y

in the form of eigenvalue decomposition

$$X = USU^T$$

$$Y = VZV^T$$

where $S = diag\{s_1, s_2, ..., s_n\}$ and $Z = diag\{z_1, z_2, ..., z_n\}$. Symmetric matrices X and Y are replaced by X_W and Y_W such that

$$X_W = U|S|U^T$$

$$Y_W = V|Z|V^T$$

where $|Z| = diag\{|z_1|, |z_2|, ..., |z_n|\}$ with $|z_1| \geq |z_2| \geq ... \geq |z_n| \geq 0$ and $|S| = diag\{|s_1|, |s_2|, ..., |s_n|\}$ with $|s_1| \geq |s_2| \geq ... \geq |s_n| \geq 0$ and . The new controllability Gramian PW and observability Gramian Q_W are obtained from following Lyapunov equations

$$AP_W + P_W A^T + B_W B_W^T = 0 \tag{2.23a}$$

$$A^T Q_W + Q_W A + C_W^T C_W = 0 \tag{2.23b}$$

where the fictitious input and output matrices B_W and C_W, respectively, are defined as $B_W = U|S|^{\frac{1}{2}}$ and $C_W = |Z|^{\frac{1}{2}}V$. The new Gramians P_W and Q_W are diagonalized by T_W such that

$$T_W^{-1} P_W T_W^{-1} = T_W^T Q_W T_W = diag\{\sigma_1, \sigma_2, ..., \sigma_n\}$$

where σ_n are the HSV's and $\sigma_i \geq sigma_{i+1}$ for $i = 1, 2, ..., n-1$.

Remark 5 *The realization $\{A, B_W, C_W\}$ is minimal and the ROM $\{A_r, B_r, C_r\} = \{A_{11}, B_1, C_1\}$ is obtained by truncating the low energy states in (2.15) and guaranteed to be stable.*

2.5 Passivity Preserving Model Order Reduction

Passivity preserving MOR is supposed to be an extension of balanced realization, which deals with both frequency weighted and un-weighted cases. Due to certain factors such as the importance of passivity preserving in MOR techniques and to limit the computational cost in SVD based methods, so for reasonable work has been done on passivity preserving MOR [25,30,24,36,27,29,26]. In this section, a review some of the passivity preserving MOR techniques available in literature related to both frequency weighted and un-weighted scenarios will be presented.

2.5.1 The Technique of Phillips et al.

Phillips et al. [25] presented a family of algorithms for passive ROMs which are similar to the well-known BT method [1] for stable ROMs. The main difference between two techniques is the way controllability Gramian and observability Gramian are computed. In BT [1] controllability Gramian and observability Gramian are computed by using the Lyapunov equations without using frequency weights, while Phillips et al. [25] computed the controllability and observability Gramians from the Lur'e equations without using frequency weights. By using BT [1] stable ROMs are obtained, while using the technique of Phillips et al. [25] passive ROMs are obtained. Positive real truncated balanced realization (PR-TBR) algorithm presented in [25] gives guaranteed passive ROM for the system $G(s) = D + C(sI - A)^{-1}B$. Controllability Gramian P_P and observability Gramian Q_P are the solutions of following Lure equations

$$AP_P + P_PA^T = -K_iK_i^T \tag{2.24a}$$

$$P_PC^T - B = -K_iW_i^T \tag{2.24b}$$

$$W_iW_i^T = D + D^T \tag{2.24c}$$

$$A^TQ_P + Q_PA = -K_o^TK_o \tag{2.25a}$$

$$Q_PB - C^T = -K_o^TW_o \tag{2.25b}$$

$$W_o^TW_o = D + D^T \tag{2.25c}$$

The above Lur'e equations can also be transformed into following ARE's

$$AP_P + P_PA^T + (P_PC^T - B)(D + D^T)^{-1}(CP_P - B^T) = 0 \tag{2.26a}$$

$$A^TQ_P + Q_PA + (Q_PB - C^T)(D + D^T)^{-1}(B^TQ_P - C) = 0 \tag{2.26b}$$

Algorithm 2 in [35] computes similarity transformation T_P which is used to diagonalized P_P and Q_P such that $T_P^{-1}P_PT_P^{-T} = T_P^TQ_PT_P = diag\{\sigma_1, \sigma_2, ..., \sigma_n\}$ where σ_n are the HSV's and $\sigma_i \geq \sigma_{i+1}$ for $i = 1, 2, ..., n-1$. Desired ROM $\{A_{11}, B_1, C_1\}$ ia obtained from the following balanced realization

$$\left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] = \left[\begin{array}{c|c} T_P^{-1}AT_P & T_P^{-1}B \\ \hline CT_P & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \tag{2.27}$$

2.5.2 Techniques of Muda et al.

In [27] Muda et al. extended three famous techniques [9, 28, 10] to preserve passivity of the ROM. Consider the input and output weighting functions in (3.13a) and (3.13b) respectively, and input and output augmented systems in (3.14) and (3.15), respectively. Let $\bar{P}i$ and $\bar{Q}_o$ be the solution of the following Lur'e equations:

$$\bar{A}_i\bar{P}_i + \bar{P}_i\bar{A}_i^T = -\bar{K}_i\bar{K}_i^T \tag{2.28a}$$

$$\bar{P}_i\bar{C}_i^T - \bar{B}_i = \bar{K}_i\bar{W}_i^T \tag{2.28b}$$

$$\bar{W}_i\bar{W}_i^T = \bar{D}_i + \bar{D}_i^T \tag{2.28c}$$

$$\bar{A}_o^T\bar{Q}_o + \bar{Q}_o\bar{A}_o = -\bar{K}_o^T\bar{K}_o \tag{2.29a}$$

$$\bar{Q}_o\bar{B}_o - \bar{C}_o^T = -\bar{K}_o^T\bar{W}_o \tag{2.29b}$$

$$\bar{W}_o^T\bar{W}_o = \bar{D}_o + \bar{D}_o^T \tag{2.29c}$$

where

$$\bar{W}_i = (\bar{D}_i + \bar{D}_i^T)^{\frac{1}{2}}V \tag{2.30a}$$

$$\bar{W}_o = U(\bar{D}_i + \bar{D}_i^T)^{\frac{1}{2}} \tag{2.30b}$$

$$\bar{K}_i = (\bar{B}_i - \bar{P}_i\bar{C}_i^T)(\bar{D}_i + \bar{D}_i^T)^{-\frac{1}{2}}V \tag{2.31a}$$

$$\bar{K}_o = U(\bar{D}_i + \bar{D}_i^T)^{-\frac{1}{2}}(\bar{C}_o - \bar{B}_o^T\bar{Q}_o) \tag{2.31b}$$

where V and U are arbitrary orthognal matrices i-e. $UU^T = VV^T = I.\bar{K}_i$ and $\bar{K}_o$ can be sub-divided as

$$\bar{K}_i = \begin{bmatrix} K_{i1} \\ K_{i2} \end{bmatrix}, \ \bar{K}_o = \begin{bmatrix} K_{o1} & K_{o2} \end{bmatrix}$$

The Lur'e equations in (2.28) and (2.29) are equivalent to the following ARE's

$$\bar{A}_i\bar{P}_i + \bar{P}_i\bar{A}_i^T + (\bar{P}_i\bar{C}_i^T - \bar{B}_i)(\bar{D}_i + \bar{D}_i^T)^{-1}(\bar{C}_i\bar{P}_i - \bar{B}_i^T) = 0 \tag{2.32a}$$

$$\bar{A}_o^T\bar{Q}_o + \bar{Q}_o\bar{A}_o + (\bar{Q}_o\bar{B}_o - \bar{C}_o^T)(\bar{D}_o + \bar{D}_o^T)^{-1}(\bar{B}_o^T\bar{Q}_o - \bar{C}_o) = 0 \tag{2.32b}$$

where

$$\bar{P}_i = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \tag{2.33a}$$

$$\bar{Q}_o = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \tag{2.33b}$$

Expanding (1,1) block of (2.28) and (2.29) we get

$$AP_{11} + P_{11}A^T = -\hat{X} \tag{2.34a}$$

$$P_{11}C^T - \tilde{B} = -K_{i1}W_i^T \tag{2.34b}$$

$$W_iW_i^T = \bar{D}_i + \bar{D}_i^T \tag{2.34c}$$

$$A^TQ_{11} + Q_{11}A = \hat{Y} \tag{2.35a}$$

$$Q_{11}B - \tilde{C}^T = -K_{o1}^TW_o \tag{2.35b}$$

$$W_o^TW_o = \bar{D}_o + \bar{D}_o^T \tag{2.35c}$$

where

$$\hat{X} = BC_iP_{12}^T + P_{12}C_i^TB^T + K_{i1}K_{i1}^T \tag{2.36a}$$

$$\hat{Y} = C^TB_o^TQ_{12}^T + Q_{12}B_oC + K_{o1}^TK_{o1} \tag{2.36b}$$

$\tilde{B} = BD_i - P_{12}C_i^TD^T$ and $\tilde{C} = D_oC - D^TB_o^TQ_{12}^T$. The difference between Muda's extended techniques [27] is that the way frequency weighted controllability and observability Gramians are computed.

2.5.3 The Modified Enns' Technique

Rewriting (2.36a) and (2.36b) as

$$\hat{X} = \psi + K_{i1}K_{i1}^T \tag{2.37a}$$

$$\hat{Y} = \phi + K_{o1}^T K_{o1} \tag{2.37b}$$

where $\psi = BC_iP^T + P_{12}C_i^T B^T$ and $\phi = C^T B_o^T Q_{12}^T + Q_{12}B_oC$. The modified Enns' technique [27] is very similar to the standard Enns' technique [9]. In standard Enns' technique [9], the indefinite matrices X and Y in (2.14a) and (2.14b) are the reason of instability in case of double sided weighting. Similarly, in modified Enns' technique [27], the indefinite matrices ψ and ϕ in (2.37a) and (2.37b) are the reason of not ensuring passivity for double sided frequency weights.

Remark 6 *The ROM obtained from modified Enns' technique [27] is passive only when the matrices ψ and ϕ are positive semi-definite. If the matrices ψ and ϕ in* (2.37a) *and* (2.37b) *are indefinite, then this technique may yields non passive ROM for double sided frequency weights.*

Remark 7 *The weighted Gramians P_{11} and Q_{11} are used to obtained transformation matrix T_E which is then used to balance the original system.*

2.5.4 The Modified Wang et al. Technique

The matrices $\hat{X}$ and $\hat{Y}$ given in (2.36a) and (2.36b) are generally indefinite [27]. Inspiring from [28], the indefinite matrices $\hat{X}$ and $\hat{Y}$ can be made positive semi definite by taking the absolute of eigenvalues of matrices $\hat{X}$ and $\hat{Y}$ by using eigenvalue decomposition such that $\hat{X} = \Delta S \Delta^T$ and $\hat{Y} = \delta Z \delta^T$ where, $Z = diag\{z_1, z_2, ..., z_n\}$ and $S = diag\{s_1, s_2, ..., s_n\}$. Now we shall replace $\hat{X}$ and $\hat{Y}$ by $\bar{K}_{i1}\bar{K}_{i1}^T$ and $\bar{K}_{o1}^T\bar{K}_{o1}$ such that $\bar{K}_{i1}\bar{K}_{i1}^T = \Delta|S|\Delta^T$, where $|S| = |s_1| \geq |s_2| \geq ... \geq |s_n| \geq 0$ and $\bar{K}_{o1}^T\bar{K}_{o1} = \delta|Z|\delta^T$ where $|Z| = |z_1| \geq |z_2| \geq ... \geq |z_n| \geq 0$. Now we can rewrite

(2.34) and (2.35) as

$$AP_{w1} + P_{w1}A^T = -\bar{K}_{i1}\bar{K}_{i1}^T \tag{2.38a}$$

$$P_{w1}C^T - \hat{B} = -\bar{K}_{i1}\bar{W}_i^T \tag{2.38b}$$

$$\bar{W}_i\bar{W}_i^T = \bar{D}_i + \bar{D}_i \tag{2.38c}$$

$$A^TQ_{w1} + Q_{w1}A = -\bar{K}_{o1}^T\bar{K}_{o1} \tag{2.39a}$$

$$Q_{w1}B - \hat{C}^T = \bar{K}_{o1}^T\bar{W}_o \tag{2.39b}$$

$$\bar{W}_o^T\bar{W}_o = \bar{D}_o + \bar{D}_o \tag{2.39c}$$

where $\hat{B} = P_{w1}C^T + \bar{K}_{i1}\bar{W}_i^T$ and $\hat{C} = B^TQ_{w1} + C^T + \bar{W}_o^T\bar{K}_{o1}$

Remark 8 *If the matrices $\hat{X}$ and $\hat{Y}$ in* (2.36a) *and* (2.36b) *respectively, are already positive semi-definite then, both modified Enns' and modified Wang et al. methods are same and guaranteed to be passive. In this case eigenvalue decomposition of $\hat{X}$ and $\hat{Y}$ is not required.*

Remark 9 *The state space realization $\{A, \hat{B}, \hat{C}\}$ is minimal and the ROM $\{A_r, B_r, C_r\} = \{A_{11}, B_1, C_1\}$ is obtained by truncating the low energy states as obtained from* (2.15) *and guaranteed to be passive.*

2.5.5 The Modified Lin and Chiu's Technique

This modified technique is similar to Lin and Chiu's technique discussed in [10]. In this technique controllability and observability Gramians of input and output augmented systems respectively, are block diagonally transformed to compute the FW original system's controllability and observability Gramians using the following transformation matrices

$$T_i = \left[\begin{array}{c|c} I & \bar{X}_i \\ \hline 0 & I \end{array}\right] \tag{2.40a}$$

$$T_o = \left[\begin{array}{c|c} I & -\bar{Y}_o \\ \hline 0 & I \end{array}\right] \tag{2.40b}$$

$$\tag{2.40c}$$

where $\hat{X}_i = P_{12}P_{22}^{-1}$ and $\hat{Y}_o = Q_{22}^{-1}Q_{12}^T$. The Gramians P_i and Q_o are obtained by transforming $\bar{P}_i$ and $\bar{Q}_o$ as follows:

$$P_i = T_i^{-1}\bar{P}_i T_i^{-T} = \left[\begin{array}{c|c} \hat{P} & 0 \\ \hline 0 & P_{22} \end{array}\right] \tag{2.41a}$$

$$Q_o = T_o^T \bar{Q}_o T_o = \left[\begin{array}{c|c} \hat{Q} & 0 \\ \hline 0 & Q_{22} \end{array}\right] \tag{2.41b}$$

where $\hat{P} = P_{11} - P_{12}P_{22}^{-1}P_{12}^T$ and $\hat{Q} = Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^T$. The transformed input and output augmented realizations $\{\tilde{A}_i, \tilde{B}_i, \tilde{C}_i, \tilde{D}_i\}$ and $\{\tilde{A}_o, \tilde{B}_o, \tilde{C}_o, \tilde{D}_o\}$ respectively, are as follows:

$$\left[\begin{array}{c|c} \tilde{A}_i & \tilde{B}_i \\ \hline \tilde{C}_i & \tilde{D}_i \end{array}\right] = \left[\begin{array}{c|c} T_i^{-1}\bar{A}_i T_i & T_i^{-1}\bar{B}_i \\ \hline \bar{C}_i T_i & \bar{D}_i \end{array}\right] = \left[\begin{array}{cc|c} A & A_{i12} & B_{i11} \\ 0 & A_i & B_i \\ \hline C & C_{i12} & \bar{D}_i \end{array}\right] \tag{2.42}$$

and

$$\left[\begin{array}{c|c} \tilde{A}_o & \tilde{B}_o \\ \hline \tilde{C}_o & \tilde{D}_o \end{array}\right] = \left[\begin{array}{c|c} T_o^{-1}\bar{A}_o T_o & T_o^{-1}\bar{B}_o \\ \hline \bar{C}_o T_o & \bar{D}_o \end{array}\right] = \left[\begin{array}{cc|c} A & A_{o12} & B \\ 0 & A_o & B_{o21} \\ \hline C_{o11} & C_o & \bar{D} \end{array}\right] \tag{2.43}$$

where

$$A_{i12} = A\hat{X}_i + BC_i - \hat{X}A_i \tag{2.44a}$$

$$B_{i11} = BD_i - \hat{X}_iB_i \tag{2.44b}$$

$$C_{i12} = C\hat{X}_i + DC_i \tag{2.44c}$$

$$A_{o12} = \hat{Y}_oA + B_oC - A_o\hat{Y}_o \tag{2.44d}$$

$$B_{o12} = B_oD + \hat{Y}_oB \tag{2.44e}$$

$$C_{o11} = D_oC - C_o\hat{Y}_o \tag{2.44f}$$

The augmented realizations satisfy the following ARE's:

$$\tilde{A}_iP_i + P_i\tilde{A}_i^T + (P_i\tilde{C}_i^T - \tilde{B}_i)(\tilde{D}_i + \tilde{D}_i^T)^{-1}(\tilde{C}_iP_i - \tilde{B}_i^T) = 0 \tag{2.45a}$$

$$\tilde{A}_o^TQ_o + Q_o\tilde{A}_o + (Q_i\tilde{B}_o - \tilde{C}_o^T)(\tilde{D}_o + \tilde{D}_o^T)^{-1}(\tilde{B}_o^TQ_o - \tilde{C}_o) = 0 \tag{2.45b}$$

Expanding the (1,1) block of (2.45a) and (2.45b) yield

$$A\hat{P} + \hat{P}A^T + (\hat{P}C^T - B_{i11})(\bar{D}_i + \bar{D}_i^T)^{-1}(C\hat{P} - B_{i11}^T) = 0 \tag{2.46a}$$

$$A^T\hat{Q} + \hat{Q}A + (\hat{Q}B - C_{o11}^T)(\bar{D}_o + \bar{D}_o^T)^{-1}(B^T\hat{Q} - C_{o11}) = 0 \tag{2.46b}$$

Remark 10 *As the FW Gramians $\hat{P}$ and $\hat{Q}$ satisfy above ARE's these Gramians also satisfy the corresponding Lur'e equation so, the ROM obtained from modified technique of Lin and Chiu's [27] is claimed to preserve passivity of ROM.*

Remark 11 *The realization $\{A, B_{i11}, C_{o11}\}$ is minimal, and the ROM is obtained by balancing and partitioning this minimal realization.*

2.5.6 The Technique of Heydari and Pedram

The technique of Heydari and Pedram [26] is the extension of Phillips et al.'s technique [25] to include the effect of frequency weights in MOR. Let $G_i(s)$ and $G_o(s)$ be the PR input and output frequency weights respectively, as given in (3.13a) and (3.13b) with state space realizations $\{A_i, B_i, C_i, D_i\}$ and $\{A_o, B_o, C_o, D_o\}$. Let the augmented systems $G(s)G_i(s)$ and $G_o(s)G(s)$ as defined in (3.14) and (3.15) respectively. Let $\bar{P}$

and $\bar{Q}$ are obtained from the following AREs

$$\bar{A}_i\bar{P} + \bar{P}\bar{A}_i^T + (\bar{P}\bar{C}_i^T - \bar{B}_i)(\bar{D}_i + \bar{D}_i^T)^{-1}(\bar{C}_i\bar{P} - \bar{B}_i^T) = 0 \tag{2.47a}$$

$$\bar{A}_o^T\bar{Q} + \bar{Q}\bar{A}_o + (\bar{Q}\bar{B}_o - \bar{C}_o^T)(\bar{D}_o + \bar{D}_o^T)^{-1}(\bar{B}_o^T\bar{Q} - \bar{C}_o) = 0 \tag{2.47b}$$

where

$$\bar{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \tag{2.48a}$$

$$\bar{Q} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \tag{2.48b}$$

Expanding the (1,1) block of (2.47a) and (2.47b) yields

$$AP_{11}+P_{11}A^T+\underbrace{BC_iP_{12}^T + P_{12}C_i^TB^T}+(P_{11}C^T+\underbrace{P_{12}C_i^TD^T}-BD_i)(DD_i+D_i^TD^T)^{-1}$$
$$(CP_{11} + \underbrace{DC_iP_{12}^T}-D_i^TB^T) = 0 \tag{2.49a}$$

$$A^TQ_{11}+Q_{11}A+\underbrace{C^TB_o^TQ_{12}^T + Q_{12}B_oC}+(Q_{11}B+\underbrace{Q_{12}B_oD}-C^TD_o^T)(C_oD+D^TD_o^T)^{-1}$$
$$(B^TQ_{11} + \underbrace{D^TB_o^TQ_{12}^T}-D_oC) = 0 \tag{2.49b}$$

In (2.47a) and (2.47b), we define the combined effect of under braced terms as $\bar{X}$ and $\bar{Y}$ respectively, where

$$\begin{aligned}\bar{X} = {} & BC_iP_{12}^T + P_{12}C_i^TB^T + P_{11}C^T(DD_i + D_i^TD^T)^{-1}DC_iP_{12}^T + \\ & P_{12}C_i^TD^T(DD_i + D_i^TD^T)^{-1}CP_{11} + P_{12}C_i^TD^T(DD_i + D_i^TD^T)^{-1}DC_iP_{12}^T - \\ & P_{12}C_i^TD^T(DD_i + D_i^TD^T)^{-1}D_i^TB^T - BD_i(DD_i + D_i^TD^T)^{-1}DC_iP_{12}^T\end{aligned} \tag{2.50a}$$

$$\begin{aligned}\bar{Y} = {} & C^TB_o^TQ_{12}^T + Q_{12}B_oC + Q_{11}B(D_oD + D^TD_o^T)^{-1}D^TB_o^TQ_{12}^T + \\ & Q_{12}B_oD(D_oD + D^TD_o^T)^{-1}B^TQ_{11} + Q_{12}B_oD(D_oD + D^TD_o^T)D^TB_o^TQ_{12}^T - \\ & Q_{12}B_oD(D_oD + D^TD_o^T)^{-1}D_oC - C^TD_o^T(D_oD + D^TD_o^T)^{-1}D^TB_o^TQ_{12}^T\end{aligned} \tag{2.50b}$$

Similar to the technique of [9] for stable ROMs, Heydari and Pedram [26] made generally indefinite symmetric matrices $\bar{X}$ and $\bar{Y}$ in (2.50a) and (2.50b) positive semi definite by taking absolute of the eigenvalues of $\bar{X}$ and $\bar{Y}$ using eigenvalue decomposition such that $\bar{X} = USU^T$ and $Y = VZV^T$, where $S = diag\{s_1, s_2, ..., s_n\}$ with $|s_1| \geq |s_2| \geq ... \geq |s_n| \geq 0$ and $Z = diag\{z_1, z_2, ..., z_n\}$, with $|z_1| \geq |z_2| \geq ... \geq |z_n| \geq 0$. Balancing transformation matrix T_H which is used to diagonalize the weighted Gramians $P_H = P_{11}$ and $Q_H = Q_{11}$ such that $T_H^{-1} P_H T_H^{-T} = T_H^T Q_H T_H = diga\{\sigma_{h1}, \sigma_{h2}, ..., \sigma_{hn}\}$ where σ_{hn} are the HSV's and $\sigma_{hi} \geq \sigma_{hi+1}$ for $i = 1, 2, ..., n - 1$. Then the ROM $\{A_r, B_r, C_r\} = \{A_{11}, B_1, C_1\}$ can be obtained as follows:

$$\left[\begin{array}{c|c} A_b & B_b \\ \hline C_b & D_b \end{array}\right] = \left[\begin{array}{c|c} T_H A T_H^{-1} & T_H B \\ \hline C T_H^{-1} & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \tag{2.51}$$

2.6 Muda et al.'s Comment on Heydari and Pedram's Technique

Muda et al. [29] proved that the technique of Heydari and Pedram [26] neither yields passive nor stable ROM in the case of double sided frequency weights. And passivity is guaranteed only for the case of single sided weighting. Let $G_i(s)$ and $G_o(s)$ be the input and output frequency weights as given in (3.13a) and (3.13b). Let the input and output augmented systems $G(s)G_i(s)$ and $G_o(s)G(s)$ respectively, as defined in (3.14) and (3.15). Let the controllability Gramian P_i and observability Gramian Q_o be the solutions of the Lur'e equations in (2.28) and (2.29), respectively. If $D = 0$, then the Lur'e equations in (2.28) and (2.29) reduces to

$$\bar{A}_i \bar{P}_i + \bar{P}_i \bar{A}_i^T = -\bar{K}_i \bar{K}_i^T \tag{2.52a}$$

$$\bar{P}_i \bar{C}_i^T = \bar{B}_i \tag{2.52b}$$

$$\bar{A}_o^T \bar{Q}_o + \bar{Q}_o \bar{A}_o = -\bar{K}_o^T \bar{K}_o \tag{2.53a}$$

$$\bar{Q}_o \bar{B}_o = \bar{C}_o^T \tag{2.53b}$$

where

$$\bar{P}_i = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \tag{2.54a}$$

$$\bar{Q}_o = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \tag{2.54b}$$

and

$$\bar{K}_i = \begin{bmatrix} K_{i1} \\ K_{i2} \end{bmatrix}, \ \bar{K}o = \begin{bmatrix} K_{o1} & K_{o2} \end{bmatrix} \tag{2.55}$$

Remark 12 *For $D = 0$ the solution of the Lur'e equations is not as simple as when $D \neq 0$. For $D = 0$, Lur'e equations can be solved by the method of [35].*

Expanding the (1,1) block of (2.52) and (2.53) yield

$$\begin{aligned} AP_{11} + P_{11}A^T &= -\hat{X} && (2.56a) \\ P_{11}C^T &= BD_i && (2.56b) \end{aligned}$$

$$\begin{aligned} A^TQ_{11} + Q_{11}A &= -\hat{Y} && (2.57a) \\ Q_{11}B &= C^TD_o^T && (2.57b) \end{aligned}$$

where $\hat{X}$ and $\hat{Y}$ are same as defined in (2.36a) and (2.36b). The matrices $\hat{X}$ and $\hat{Y}$ are generally indefinite. To ensure passivity, these matrices should be positive semi-definite. This can be accomplished by eigenvalue decomposition such that $\hat{X} = \Delta S \Delta^T$ and $Y = \delta Z \delta^T$ where, $Z = diag\{z_1, z_2, ..., z_n\}$ and $S = diag\{s_1, s_2, ..., s_n\}$. Now we shall replace $\hat{X}$ and $\hat{Y}$ by $\bar{K}_{i1}\bar{K}_{i1}^T$ and $\bar{K}_{o1}^T\bar{K}_{o1}$ such that $\bar{K}_{i1}\bar{K}_{i1}^T = \Delta|S|\Delta^T$ where where $|S| = |s_1| \geq |s_2| \geq ... \geq |s_n| \geq 0$ and $\bar{K}_{o1}^T\bar{K}_{o1} = \delta|Z|\delta^T$ where $Z = |z_1| \geq |z_2| \geq ... \geq |z_n| \geq 0$. Now we can rewrite (2.56) and (2.57) as

$$\begin{aligned} AP_h + P_hA^T &= \bar{K}_{i1}\bar{K}_{i1}^T && (2.58a) \\ P_hC^T &= \hat{B} && (2.58b) \end{aligned}$$

$$A^TQ_h + Q_hA = \bar{K}_{o1}^T\bar{K}_{o1} \quad (2.59a)$$

$$Q_hB = \hat{C}^T \quad (2.59b)$$

Transformation matrix $\hat{T}$ is used for diagonalizing the Gramians P_h and Q_h, i.e. $\hat{T}^{-1}P_h\hat{T}^{-T} = \hat{T}^TQ_h\hat{T} = diag\{\sigma_1, \sigma_2, ..., sigma_n\}$ where σ_n are the HSV's and $\sigma_i \geq \sigma_{i+1}$ for $i = 1, 2, ..., n-1$. Then the ROM $\{A_{11}, B_1, C_1\}$ can be obtained from the following balanced realization.

$$\left[\begin{array}{c|c} A_b & B_b \\ \hline C_b & D_b \end{array}\right] = \left[\begin{array}{c|c} \hat{T}^{-1}AT & \hat{T}_{-1}B \\ \hline C\hat{T} & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \quad (2.60)$$

Remark 13 *The ROM obtained from Heydari and Pedram [26] is not passive because* (2.58a)*,*(2.58b) *and* (2.59a)*,*(2.59b) *don't correspond to the same system but two different systems* $\{A, \hat{B}, C\}$ *and* $\{A, B, \hat{C}\}$ *respectively.*

Remark 14 *The transformation matrix* $\hat{T}$*, which is obtained by diagonalizing the Gramians of two different systems* $\{A, \hat{B}, C\}$ *and* $\{A, B, \hat{C}\}$*, applied to the original system* $\{A, B, C\}$ *does not produce passive ROM.*

2.7 Summary

The existing techniques discussed in this chapter are summarized as: The ROM obtained from BT [1] is guaranteed stable but without the presence of frequency weights. Enn's technique [9] does not yield guaranteed stable ROM in case of double sided frequency weights. The techniques of Lin and Chiu [10] and Wang et al. [28] produce guaranteed stable ROM's when double sided frequency weights are used. Although, Lin and Chiu's method [10] produces guaranteed stable ROM but due to pole-zero cancellation of the controller with the frequency weights, this technique can not be used in the controller reduction applications. Wang et al.'s method [28] gives frequency response error bound as well. The technique of Phillips et al. [25] gives passive ROM without the presence of frequency weights while the technique of Muda et al. [27] gives passive ROM in the presence of frequency weights. Heydari and Pedram's [26] technique does not guarantee passivity in case of double sided weights [29].

2.8 Conclusion

First of all we discussed the important properties of BT and frequency weighted MOR. Then we formulated together frequency weighted and passivity preserving MOR problem and also discussed the important properties of frequency weighted passivity preserving MOR. Several frequency weighted and un-weighted MOR techniques were discussed and their important characteristics were also presented. We can conclude that, like un-weighted MOR, preserving stability and passivity of the ROM as well as low approximation error are also desirable in the frequency weighted MOR.

Chapter 3

Development of New Passivity-Preserving Model Order Reduction Schemes

3.1 Introduction

The design of any system e.g. network and telecommunication systems contain mathematical model which are complex and require lot of computational power for analysis and simulation. For such scenarios, to make the analysis and design of such systems easier, MOR is used which reduce the complexity of such models by reducing their order [23]. It takes less computational power and simulation time when models with reduced orders are used. MOR methods are very useful when it comes to analyze large and complex systems such as space systems, state estimation in UAVs and high voltage systems etc.

In feedback control systems theory, the major contribution of balanced truncation is its application in MOR which gives the stable reduced models and an error bound within certain frequency limit [28]. This is also known as frequency weighted MOR problem. For a given transfer function there are almost infinite state space realizations but a particular realization has been proved useful in control systems theory which is called internally balanced realization. It indicates the dominant system states and it is the minimal realization which is also asymptotically stable.

The most wanted property in some systems is passivity which can be implied if the transfer function is positive real. And when the model is reduced to rth ROM using MOR it is desired to preserve the important features such as passivity, stability and error. Since passive systems are also stable systems and not vice versa, it is of vital importance in MOR algorithms to preserve passivity [24].

Phillips et al. [25] presented a family of algorithms for passive ROM which are similar to the well-known BT method [1] for stable ROMs. In [25] controllability and observability Gramians are obtained by solving the Lur'e equations without using frequency weights. Muda et al. [27] extended the methods of [28], [10] and [9] for RLC systems to ensure passivity, since [28], [10], [9] yield only stable ROMs. Conditions for guaranteed passivity are also given in [27] for the three extended techniques. A relationship between the Lur'e equations and the algebraic Riccati equations (ARE's) is also given in [35]. For $D = 0$ in a state space realization $\{A, B, C, D\}$ the solution of an ARE is not straight forward. Both cases, for $D = 0$ and $D \neq 0$, are discussed in detail in [35].

Heydari and Pedram claimed in [26] that their technique produces guaranteed passive ROM for the double sided frequency weighting case, and the spectrally-weighted error bounds are also available. The technique of [26] produces passive ROM for the single sided weighting case similar to [9] which produces stable ROM for the case when only one sided weighting is used. In [29] it has been proved that the technique of [26] may yields non-passive ROM for the passive original system in case of double-sided weighting. [26] also proved that the method of [26] can preserve passivity only when one sided weighting is present.

After that, a lot of work has been done on preserving passivity in case of double sided frequency weightings [24], [30], [36] for CT time systems. But currently there hasn't been any work done for passivity preserving schemes for DT linear time invariant systems. However, [31] proposed MOR scheme for DT linear time variant macromodels. In this chapter a family of schemes are proposed for passivity preserving for LTI DT systems. Error bounds are also derived to prove the efficiency of the proposed techniques.

3.2 Preliminaries

Consider a linear time invariant discrete time (DT) passive system:

$$x[n+1] = Ax[n] + Bu[n] \tag{3.1a}$$

$$y[n] = Cx[n] + Du[n] \tag{3.1b}$$

where $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{k\times n}$ and $D \in \Re^{k\times m}$ are the state space representation of the discrete system. The transfer function form is given as:

$$H(z) = C(zI - A)^{-1}B + D \tag{3.2}$$

Let the system satisfy following DT Lyapunov equations [12]

$$AP_U A^T - P_U + BB^T = 0 \tag{3.3}$$

$$A^T Q_U A - Q_U + C^T C = 0 \tag{3.4}$$

and Lur'e equations [37]:

$$AP_{LU}A^T - P_{LU} = -HH^T \tag{3.5a}$$

$$AP_{LU}C^T - B = -HJ^T \tag{3.5b}$$

$$D + D^T - CP_{LU}C^T = JJ^T \tag{3.5c}$$

$$A^T Q_{LU} A - Q_{LU} = -K^T K \tag{3.6a}$$

$$A^T Q_{LU} B - C^T = -K^T L \tag{3.6b}$$

$$D + D^T - B^T Q_{LU} B = L^T L \tag{3.6c}$$

The matrices $P_{LU} \in \Re^{n\times n}$ and $Q_{LU} \in \Re^{n\times n}$ can also be computed [38] using discrete algebraic Riccati equations (DAREs):

$$AP_{LU}A^T - P_{LU} + (B - AP_{LU}C^T)(D + D^T - CP_{LU}C^T)^{-1}(B^T - CP_{LU}A^T) = 0$$

$$A^T Q_{LU} A - Q_{LU} + (C^T - A^T Q_{LU} B)(D + D^T - B^T Q_{LU} B)^{-1}(C - B^T Q_{LU} A) = 0$$

3.3 Unweighted Model Reduction

Let $P_i = \{P_U, P_{LU}\}$ and $Q_j = \{Q_U, Q_{LU}\}$ and the transformation matrix T_{ij} can be computed by diagonalizing P_i and Q_i as:

$$T_{ij}^{-1} P_i T_{ij}^{-T} = T_{ij}^T Q_j T_{ij} = \Sigma_d = diag\{\sigma_1, \sigma_2, ..., \sigma_n\} \tag{3.7}$$

where $i = 1, 2$ and $j = 1, 2$. Let the original system is transformed as following:

$$\left[\begin{array}{c|c} T_{ij}^{-1} A T_{ij} & T_{ij} B \\ \hline C T_{ij} & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \tag{3.8}$$

The reduced order model is $\{A_{11}, B_1, C_1, D\}$.

Theorem 1 *For the case when* $\{i, j\} = \{1, 2\}, \{2, 1\}, \{2, 2\}$ *the ROM obtained from* (3.8) *is passive.*

Proof: For case when $\{i, j\} = \{2, 2\}$, let Σ_d from (3.7) can be partitioned as:

$$\Sigma_d = \begin{bmatrix} \Sigma_{d1} & 0 \\ 0 & \Sigma_{d2} \end{bmatrix}$$

After similar partition of $K = \begin{bmatrix} K_{d_1} & K_{d_2} \end{bmatrix}$ and expanding (1,1) block of (3.6) we get:

$$A_{11}^T \Sigma_{d_1} A_{11} - \Sigma_{d_1} + A_{21}^T \Sigma_{d_2} A_{21} = -K_{d_1}^T K_{d_1} \tag{3.9a}$$

$$A_{11}^T \Sigma_{d_1} B_1 + A_{21} \Sigma_{d_2} B_2 - C_1^T = -K_{d_1}^T L \tag{3.9b}$$

$$D + D^T - B_1^T \Sigma_{d_1} B_1 - B_2^T \Sigma_{d_2} B_2 = L^T L \tag{3.9c}$$

In CT case due to the structure of Lur'e equation we don't have extra terms like $A_{21}^T \Sigma_{d_2} A_{21}$ and $A_{21}^T \Sigma_{d_2} B_2$ as in DT case, and the sub-block expansion readily satisfies the standard equation's format. But analogous to lemma 2.2 and theorem 2.1 of [39] we can show that these extra terms don't affect the passivity of ROM. Let Σ' be the observability Gramian of ROM that satisfy following equations, we have $\Sigma' \geq 0$.

$$A_{11}^T \Sigma' A_{11} - \Sigma' = -K_{d_1}^T K_{d_1} \tag{3.10a}$$

$$A_{11}^T \Sigma' B_1 - C_1^T = -K_{d_1}^T L \tag{3.10b}$$

$$D + D^T - B_1^T \Sigma' B_1 = L^T L \tag{3.10c}$$

By linear system DT positive real lemma there exists a matrix $\Omega \geq 0$ such that

$$A_{11}^T \Omega A_{11} - \Omega = -A_{21}^T \Sigma_{d_2} A_{21} \tag{3.11a}$$

$$A_{11}^T \Omega B_1 = -A_{21}^T \Sigma_{d_2} B_2 \tag{3.11b}$$

$$-B_1^T \Omega B_1 = B_2^T \Sigma_{d_2} B_2 \tag{3.11c}$$

Subtraction of (3.10) from (3.9) yields (3.11), where $\Sigma_{d1} - \Sigma' = \Omega \geq 0$ and $A_{21}^T \Sigma_{d_2} A_{21} \geq 0$. The equivalent DARE is given as:

$$A_{11}^T \Omega A_{11} - \Omega + A_{11}^T \Omega B_1 (-B_1^T \Omega B_1)^{-1} B_1^T \Omega A_{11} = 0 \tag{3.12}$$

It can also be shown for dual set of Lur'e equations.

For case $\{i, j\} = \{1, 2\}$ the ROM is passive due to existence of $\Omega \geq 0$ in (3.11) and (3.12).

Similarly for case $\{i, j\} = \{2, 1\}$ same property for ROM holds.

Remark 15 *In [29], it is discussed for CT frequency weighted case that [26] doesn't generate passive ROM because in equations (18) and (19) of [29] there are extra terms which may not be positive definite, however in proposed DT unweighted scheme these extra terms are positive definite, hence passivity is preserved.*

Remark 16 *For case $\{i, j\} = \{1, 1\}$ the ROM is stable [1].*

3.4 Frequency Weighted Model Reduction

Consider a DT transfer function as in (3.2) and the weighting functions

$$V(z) = C_v(zI - A_v)^{-1} B_v + D_v \tag{3.13a}$$

$$W(z) = C_w(zI - A_w)^{-1} B_w + D_w \tag{3.13b}$$

where A_v, B_v, C_v, D_v and A_w, B_w, C_w, D_w are their p^{th} and q^{th} order minimal realizations, respectively. Let the input augmented system be given by:

$$H(z)V(z) = \left[\begin{array}{c|c} \bar{A}_v & \bar{B}_v \\ \hline \bar{C}_v & \bar{D}v \end{array}\right] = \left[\begin{array}{cc|c} A & BC_v & BD_v \\ 0 & A_v & B_v \\ \hline C & DC_v & DD_v \end{array}\right] \tag{3.14}$$

and the output augmented matrix :

$$W(z)H(z) = \left[\begin{array}{c|c} \bar{A}_w & \bar{B}_w \\ \hline \bar{C}_w & \bar{D}_w \end{array}\right] = \left[\begin{array}{cc|c} A & 0 & B \\ B_w C & A_w & B_w D \\ \hline D_w C & C_w & D_w D \end{array}\right] \tag{3.15}$$

Let the augmented systems satisfy following DT Lyapunov Equations:

$$\bar{A}_v \bar{P} \bar{A}_v^T - \bar{P} + \bar{B}_v \bar{B}_v^T = 0 \tag{3.16a}$$

$$\bar{A}_w^T \bar{Q} \bar{A}_w - \bar{Q} + \bar{C}_w^T \bar{C}_w^T = 0 \tag{3.16b}$$

and DT Lur'e equations

$$\bar{A}_v \hat{P} \bar{A}_v^T - \hat{P} = -\hat{H}\hat{H}^T \tag{3.17a}$$

$$\bar{A}_v \hat{P} \bar{C}_v^T - \bar{B}_v = -\hat{H}\hat{J}^T \tag{3.17b}$$

$$\bar{D}_v + \bar{D}_v^T - \bar{C}_v \hat{P} \bar{C}_v^T = \hat{J}\hat{J}^T \tag{3.17c}$$

$$\bar{A}_w^T \hat{Q} \bar{A}_w - \hat{Q} = -\hat{K}^T \hat{K} \tag{3.18a}$$

$$\bar{A}_w^T \hat{Q} \bar{B}_w - \bar{C}_w^T = -\hat{K}^T \hat{L} \tag{3.18b}$$

$$\bar{D}_w + \bar{D}_w^T - \bar{B}_w^T \hat{Q} \bar{B}_w = \hat{L}^T \hat{L} \tag{3.18c}$$

The matrices $\hat{P}$ and $\hat{Q}$ can also be computed [38] using the DAREs:

$$\bar{A}_v \hat{P} \bar{A}_v^T - \hat{P} + (\bar{B}_v - \bar{A}_v \hat{P} \bar{C}_v^T)(\bar{D}_v + \bar{D}_v^T - \bar{C}_v \hat{P} \bar{C}_v^T)^{-1}(\bar{B}_v^T - \bar{C}_v \hat{P} \bar{A}_v^T) = 0 \tag{3.19}$$

$$\bar{A}_w^T \hat{Q} \bar{A}_w - \hat{Q} + (\bar{C}_w^T - \bar{A}_w^T \hat{Q} \bar{B}_w)(\bar{D}_w + \bar{D}_w^T - \bar{B}_w^T \hat{Q} \bar{B}_w)^{-1}(\bar{C}_w - \bar{B}_w^T \hat{Q} \bar{A}_w) = 0 \tag{3.20}$$

Let matrices $\bar{P}, \hat{P}, \hat{H}$ and $\bar{Q}, \hat{Q}, \hat{K}$ can be subdivided as follows:

$$\bar{P} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{12}^T & \bar{P}_{22} \end{bmatrix}, \; \bar{Q} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12}^T \\ \bar{Q}_{12} & \bar{Q}_{22} \end{bmatrix} \tag{3.21a}$$

$$\hat{P} = \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{12}^T & \hat{P}_{22} \end{bmatrix}, \; \hat{Q} = \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \hat{Q}_{12}^T & \hat{Q}_{22} \end{bmatrix} \tag{3.21b}$$

and

$$\hat{H} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \; \hat{K} = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \tag{3.22}$$

Expanding the (1,1) block of (3.16)

$$A\bar{P}_{11}A^T - \bar{P}_{11} + A\bar{P}_{12}C_v^T B^T + BC_v\bar{P}_{12}^T A^T + BC_v\bar{P}_{22}C_v^T B^T + BD_vD_v^T B^T = 0$$

$$A^T\bar{Q}_{11}A - \bar{Q}_{11} + C^T B_w^T \bar{Q}_{12}^T A + A^T\bar{Q}_{12}B_wC + C^T B_w^T\bar{Q}_{22}B_wC + C^T D_w^T D_wC = 0$$

Similarly expanding (1,1) blocks of (3.17) and (3.18) yields

$$A\hat{P}_{11}A^T - \hat{P}_{11} = -X_L \tag{3.23a}$$

$$A\hat{P}_{11}C^T - B_p = -H_1\hat{J}^T \tag{3.23b}$$

$$\bar{D}_v + \bar{D}_v - \bar{C}_v\hat{P}\bar{C}_v^T = \hat{J}\hat{J}^T \tag{3.23c}$$

$$A^T\hat{Q}_{11}A - \hat{Q}_{11} = -Y_L \tag{3.24a}$$

$$A^T\hat{Q}_{11}B - C_p^T = -K_1^T\hat{L} \tag{3.24b}$$

$$\bar{D}_w + \bar{D}_w - \bar{B}_w^T\hat{Q}\bar{B}_w = \hat{L}^T\hat{L} \tag{3.24c}$$

where

$$X_L = BC_v\hat{P}_{12}^T A^T + A\hat{P}_{12}C_v^T B^T + BC_v\hat{P}_{22}C_v^T B^T + H_1 H_1^T \quad (3.25a)$$

$$Y_L = C^T B_w^T \hat{Q}_{12}^T A + A^T\hat{Q}_{12}B_w C + C^T B_w^T\hat{Q}_{22}B_w C + K_1^T K_1 \quad (3.25b)$$

$$B_p = BD_v - BC_v\hat{P}_{12}^T C^T - A\hat{P}_{12}C_v^T D^T - BC_v\hat{P}_{22}C_v^T D^T \quad (3.25c)$$

$$C_p = D_w C - B^T\hat{Q}_{12}B_w C - D^T B_w^T\hat{Q}_{12}^T A - D^T B_w^T\hat{Q}_{22}^T B_w C \quad (3.25d)$$

Let $\tilde{P}_{\hat{i}} = \{P_U, \bar{P}_{11}, P_{LU}, \hat{P}_{11}\}$ and $\tilde{Q}_{\hat{j}} = \{Q_U, \bar{Q}_{11}, Q_{LU}, \hat{Q}_{11}\}$ and the transformation matrix $\hat{T}_{\hat{i}\hat{j}}$ can be found by diagonalizing $\tilde{P}_{\hat{i}}$ and $\tilde{Q}_{\hat{j}}$ as follows:

$$\hat{T}_{\hat{i}\hat{j}}^{-1}\tilde{P}_{\hat{i}}\hat{T}_{\hat{i}\hat{j}}^{-T} = \hat{T}_{\hat{i}\hat{j}}^T\tilde{Q}_{\hat{j}}\hat{T}_{\hat{i}\hat{j}} = \bar{\Sigma} = diag\{\bar{\sigma}_1, \bar{\sigma}_2, ..., \bar{\sigma}_n\} \quad (3.26)$$

where $\hat{i} = 1, 2, 3, 4$ and $\hat{j} = 1, 2, 3, 4$. Let the original system is transformed as following:

$$\left[\begin{array}{c|c} \hat{T}_{\hat{i}\hat{j}}^{-1}A\hat{T}_{\hat{i}\hat{j}} & \hat{T}_{\hat{i}\hat{j}}B \\ \hline C\hat{T}_{\hat{i}\hat{j}} & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \quad (3.27)$$

The reduced order model is represented by $\{A_{11}, B_1, C_1, D\}$. The properties of ROM

Table 3.1: Properties of Model reduction schemes

$\tilde{P}_{\hat{i}}$ \ $\tilde{Q}_{\hat{j}}$	Q_U	$\bar{Q}_{11}$	Q_{LU}	$\hat{Q}_{11}$
P_U	S	S	P	S
$\bar{P}_{11}$	S	S?	P	S?
P_{LU}	P	P	P	P
$\hat{P}_{11}$	S	S?	P	S?

schemes are summarized in Table 3.1 where S, P, S? respectively represent stability, passivity and stability not guaranteed.

Remark 17 *For the case when* $\{\hat{i}, \hat{j}\} = \{1, 1\}, \{3, 3\}$ *the MOR schemes are unweighted. When* $\{\hat{i}, \hat{j}\} = \{2, 2\}, \{2, 4\}, \{4, 2\}\{4, 4\}$ *the MOR schemes are double sided frequency weighted whereas for other cases when* $\{\hat{i}, \hat{j}\} = \{1, 2\}, \{2, 1\}, \{1, 3\}, \{3, 1\},$

$\{1,4\},\{4,1\},\{2,3\},\{3,2\}$,$\{3,4\},\{4,3\}$ *the MOR schemes are single sided frequency weighted.*

Theorem 2 *For the case when* $\{\hat{i},\hat{j}\} = \{1,3\},\{3,1\},\{2,3\},\{3,2\},\{3,3\},\{3,4\},\{4,3\}$ *the ROM obtained from* (3.27) *is passive.*

Proof: For the cases when $\{\hat{i},\hat{j}\} = \{1,3\},\{3,1\},\{3,3\}$ proof is already given in theorem 1. For $\{\hat{i},\hat{j}\} = \{2,3\},\{3,2\},\{3,4\},\{4,3\}$, since on one side the Gramian is computed using weighted Lur'e equations, the passivity is preserved in ROM.

Remark 18 *For the case when* $\{\hat{i},\hat{j}\} = \{2,2\},\{2,4\},\{4,2\},\{4,4\}$ *the ROMs obtained using* (3.27) *are not guaranteed to be stable.*

3.5 Proposed Scheme for Double Sided Frequency Weightings

In order preserve passivity in case of double sided frequency weightings, a scheme is suggested that divides the double sided FW problem into succession of two single sided weighted ones. At 1^{st} stage, least energy states are truncated within the frequency region that are dominated by input frequency weight and then the least energy states are truncated within frequency region that are dominated by output frequency weight in the 2^{nd} stage. The main advantage of this staging concept is that the ROM obtained is guaranteed to be passive because single sided frequency weights always generate passive ROM.

The controllability and observability Gramians are calculated from input and augmented systems respectfully which means that the passivity of ROM is dependent on the passivity of augmented system. If the augmented system is non-passive it fails to produce passive ROM, which is exactly like the cases $\{\hat{i},\hat{j}\} = \{2,2\},\{2,4\},\{4,2\},\{4,4\}$. Inspired by Umair et. al. [36] a scheme is suggested to compute the FW Gramians from discrete time Lyapunov equations and unweighted Gramian like matrices using the discrete time Lur'e equations. This methods removes the passivity condition for augmented system.

The Gramian P_{ci} of input augmented system $H(z)V(z)$ mentioned in (3.14), is called as controllability Gramian and is obtained as a solution of following DT Lyapunov equation:

$$A_iP_{ci}A_i^T - P_{ci} + B_iB_i^T = 0 \tag{3.28}$$

P_{ci} can be partitioned as:

$$P_{ci} = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{12}^T & P_{22} \end{array}\right]$$

Expanding (1,1) block of equation (3.28) we get:

$$AP_{11}A^T - P_{11} + X_1 = 0 \tag{3.29}$$

where

$$X_1 = BC_VP_{12}^TA^T + AP_{12}C_V^TB^T + BC_VP_{22}C_V^TB^T + BD_VD_V^TB^T$$

The Gramian P_h, called as FW controllability Gramian, is obtained as a solution of following DT Lyapunov equation:

$$AP_hA^T - P_h + B_hB_h^T = 0 \tag{3.30}$$

where $B_h = \bar{U}_1\bar{S}_1^{\frac{1}{2}}$ and matrices $\bar{U}_1$ and $\bar{S}_1$ are results of eigenvalue decomposition of X_1.

$$X_1 = \bar{U}\bar{S}\bar{U}^T = \left[\begin{array}{cc} \bar{U}_1 & \bar{U}_2 \end{array}\right]\left[\begin{array}{cc} \bar{S}_1 & 0 \\ 0 & \bar{S}_2 \end{array}\right]\left[\begin{array}{c} \bar{U}_1^T \\ \bar{U}_2^T \end{array}\right]$$

where $S_1 = diag\{\bar{s}_1, \bar{s}_2, ..., \bar{s}_l\}, \bar{s}_1 \geq \bar{s}_2 \geq ...\bar{s}_l > 0$ and l being the number of positive eigenvalues of X_1. Let $Y_1 = K^TK$ where the matrix K is defined in (3.6a). Then disintegration of eigenvalues of Y_1 as $Y_1 = \bar{V}\bar{R}\bar{V}^T$ where the matrix $\bar{R}$ is expressed as $\bar{R} = diag\{\bar{r}_1, \bar{r}_2, ..., \bar{r}_n\}$. The fictitious output matrix C_f is defined as $C_f = \bar{R}^{\frac{1}{2}}\bar{V}^T$. Q as mentioned in equation (3.6a) is written as a solution of following DT Lyapunov equation:

$$A^TQA - Q + C_f^TC_f = 0 \tag{3.31}$$

Q and P_h are the observability and controllability Gramian respectively of system $\{A, B_h, C_f, D\}$. The transformation matrix T_1 is calculated such that $T_1^TQT_1 = T_1^{-1}P_hT^{-T} = \bar{\Sigma} = diag\{\bar{\sigma}_1, \bar{\sigma}_2, ..., \bar{\sigma}_n\}$ and $\bar{\sigma}_1 \geq \bar{\sigma}_2 \geq ... \geq \bar{\sigma}_n$. The transformed system expressed as $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\} = \{T_1^{-1}AT_1, T_1^{-1}B, CT_1, D\}$ is then given by

$$\left[\begin{array}{c|c} T_1^{-1}AT_1 & T_1^{-1}B \\ \hline CT_1 & D \end{array}\right] = \left[\begin{array}{cc|c} A_{r_1} & A_{12} & B_{r_1} \\ A_{21} & A_{22} & B_2 \\ \hline C_{r_1} & C_2 & D \end{array}\right] \tag{3.32}$$

Then $H_{r1}(z) = C_{r_1}(zI - A_{r_1}^{-1})B_{r_1} + D$ is the $r_1^{th}(r_1 < r)$ order ROM such that $A_{r_1} \in \Re^{r_1 \times r_1}, B_{r_1} \in \Re^{r_1 \times m}$ and $C_{r_1} \in \Re^{m \times r_1}$.

Theorem 3 *The following is a priori error bound that holds for this stage of reduction if* $rank\,[B_h\ B]$ =$rank\,[B_h]$ *and* $rank \begin{bmatrix} C_f \\ C \end{bmatrix}$ = $rank\,[C_f]$*:*

$$||(H(z) - H_{r_1}(z)V(z))||_\infty \leq 2||L_1||_\infty ||K_1 V(z)||_\infty \sum_{k=r_1+1}^{n} \bar{\sigma}_k$$

where

$$L_1 = C\bar{V}diag\{\bar{r}_1^{-\frac{1}{2}}, \bar{r}_2^{-\frac{1}{2}}, ..., \bar{r}_{\bar{j}}^{-\frac{1}{2}}, 0, ..., 0\}$$

$$K_1 = diag\{\bar{s}_1^{-\frac{1}{2}}, \bar{s}_2^{-\frac{1}{2}}, ..., \bar{s}_{\bar{i}}^{-\frac{1}{2}}, 0, ..., 0\}\bar{U}^T B$$

$\bar{j} = rank\,[Y_1]$ and $\bar{i} = rank\,[X_1]$.

Proof: If $rank\,[B_h\ B]$ =$rank\,[B_h]$ then we can write following:

$$B = B_h K_1 \tag{3.33}$$

Furthermore, lets assume that $\zeta = \{K_2 | B = B_h K_3\}$ then

$$||K_1 V(z)||_\infty = \min ||K_3 V(z)||_\infty, K_3 \in \zeta$$

Since $rank\,[B_h\ B]$ =$rank\,[B_h]$ then we can parameterize solutions of $B = B_h K_3$ by following

$$K_3 = B_h^+ B + (I - B_h^+ B_h)Z \tag{3.34}$$

where B_h^+ is pseudo-inverse of B_h and Z is free parameter. Since

$$B_h^+ = diag\{\bar{s}_1^{-\frac{1}{2}}, \bar{s}_2^{-\frac{1}{2}}, ..., \bar{s}_{\bar{i}}^{-\frac{1}{2}}, 0, ..., 0\}\bar{U}^T$$

K_1 is solution of $B = B_h K_2$ when $Z = 0$ and $B = B_h K_1$. After putting expressions of B_h and B_h^+ in (3.34) we get

$$\begin{aligned} K_3 &= B_h^+ B + (I - B_h^+ B_h)Z \\ &= diag\{\bar{s}_1^{-\frac{1}{2}}, \bar{s}_2^{-\frac{1}{2}}, ..., \bar{s}_{\bar{i}}^{-\frac{1}{2}}, 0, ..., 0\}\bar{U}^T B + (I - diag(1, ..., 1, 0, ..., 0))Z \\ &= diag\{\bar{s}_1^{-\frac{1}{2}}, \bar{s}_2^{-\frac{1}{2}}, ..., \bar{s}_{\bar{i}}^{-\frac{1}{2}}, 0, ..., 0\}\bar{U}^T B + diag(0, ...0, 1, ...1)Z. \end{aligned} \quad (3.35)$$

As we have seen in previous equations that term with free matrix Z is zero, we obtain $||K_1 V(z)||_\infty \leq ||K_3 V(z)||_\infty$ for any free matrix Z which also implies that

$$||K_1 V(z)||_\infty = \min ||K_3 V(z)||_\infty, K_3 \in \zeta.$$

Similarly we can prove for $C = L_1 C_f$.

Proposition 1: ROM $\{A_{r_1}, B_{r_1}, C_{r_1}, D\}$ is passive.

Proof: We can prove that the ROM $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ is because it is obtained by the similarity transformation of $\{A, B, C, D\}$ and hence, it satisfies the following discrete time Lur'e equation provided $\bar{\Sigma} \geq 0$.

$$\bar{A}^T \bar{\Sigma} \bar{A} - \bar{\Sigma} = -\bar{K}^T \bar{K} \quad (3.36a)$$

$$\bar{A}^T \bar{\Sigma} \bar{B} - \bar{C}^T = -\bar{K}^T \bar{L} \quad (3.36b)$$

$$\bar{D} + \bar{D}^T - \bar{B}^T \bar{\Sigma} \bar{B} = \bar{L}^T \bar{L} \quad (3.36c)$$

$\bar{\Sigma}$ is divided as $diag\{\Sigma_{r_1}, \Sigma_{(n-r_1)}\}$ where Σ_{r_1} can be expressed as $\Sigma_{r_1} = diag\{\bar{\sigma}_1 \geq \bar{\sigma}_2 ... \geq \bar{\sigma}_{r_1}\}$. Expanding (1,1) block of equations (3.36a-3.36c) we get:

$$A_{r_1}^T \Sigma_{r_1} A_{r_1} - \Sigma_{r_1} + A_{21}^T \Sigma_{(n-r_1)} A_{21} = -K_1^T K_1 \quad (3.37a)$$

$$A_{r_1}^T \Sigma_{r_1} B_{r_1} + A_{21}^T \Sigma_{(n-r_1)} B_2 - C_{r_1}^T = -K_1^T L \quad (3.37b)$$

$$D + D^T - \bar{B}^T \bar{\Sigma} \bar{B} = \bar{L}^T \bar{L} \quad (3.37c)$$

Rest of proof is same as that of Theorem 1.

Consider the output augmented system $W(z)H_{r_1}(z)$ which is represented as:

$$W(z)H(z) = \left[\begin{array}{c|c} \tilde{A}_o & \tilde{B}_o \\ \hline \tilde{C}_o & \tilde{D}_o \end{array}\right] = \left[\begin{array}{cc|c} A_{r_1} & 0 & B_{r_1} \\ B_w C_{r_1} & A_w & B_w D \\ \hline D_w C_{r_1} & C_w & D_w D \end{array}\right]$$

The Gramian Q_{oo} of output augmented system $W(z)H_{r_1}(z)$ is called as observability Gramian and is obtained as a solution of following DT Lyapunov equation:

$$\tilde{A}_o^T Q_{oo} \tilde{A}_o - Q_{oo} + \tilde{C}_o^T \tilde{C}_o = 0 \tag{3.38}$$

Q_{oo} can be partitioned as:

$$Q_{oo} = \left[\begin{array}{c|c} Q_{11} & Q_{12} \\ \hline Q_{12}^T & Q_{22} \end{array}\right]$$

Expanding (1,1) block of equation (3.38) we get:

$$A_{r_1}^T Q_{11} A_{r_1} - Q_{11} + Y_2 = 0 \tag{3.39}$$

where

$$Y_2 = C_{r_1}^T B_w^T Q_{12}^T A_{r_1} + A_{r_1}^T Q_{12} B_w C_{r_1} + C_{r_1}^T B_w^T Q_{22} B_w C_{r_1} + C_{r_1}^T D_w^T D_w C_{r_1} = 0$$

The weighted Gramian Q_h called as observability Gramian, is obtained as a solution of following DT Lyapunov equation:

$$A_{r_1}^T Q_h A_{r_1} - Qh + C_h^T C_h = 0 \tag{3.40}$$

where $C_h = \bar{R}_1^{\frac{1}{2}} \bar{V}_1^T$. $\bar{R}_1$ and $\bar{V}_1$ are obtained using eigenvalue disintegration of symmetric indefinite matrix $Y2$.

$$Y_2 = \bar{V}\bar{R}\bar{V}^T = \begin{bmatrix} \bar{V}_1 & \bar{V}_2 \end{bmatrix} \begin{bmatrix} \bar{R}_1 & 0 \\ 0 & \bar{R}_2 \end{bmatrix} \begin{bmatrix} \bar{V}_1^T \\ \bar{V}_2^T \end{bmatrix}$$

where R_1 is expressed as $R_1 = diag\{\bar{r}_1, \bar{r}_2, ..., \bar{r}_{\bar{l}}\}, \bar{r}_1 \geq \bar{r}_2 \geq ...\bar{r}_{\bar{l}} > 0$ and number of positive eigenvalues of Y_2 is denoted by $\bar{l}$.The controllability Gramian like matrix $\tilde{P}_a$

of system $H_{r_1}(z)$ is solution of following DT Lur'e equation:

$$A_{r_1}\tilde{P}_a A_{r_1}^T - \tilde{P}_a = -\tilde{H}\tilde{H}^T \tag{3.41a}$$

$$A_{r_1}\tilde{P}_a C_{r_1}^T - B_{r_1} = -\tilde{H}\tilde{J}^T \tag{3.41b}$$

$$D + D^T - C_{r_1}\tilde{P}C_{r_1}^T = \tilde{J}\tilde{J}^T \tag{3.41c}$$

Similar to previous stage, let $X_2 = \tilde{H}\tilde{H}^T$. Then the eigenvalue disintegration of X_2 is $X_2 = \tilde{U}\tilde{S}\tilde{U}^T$ where $\tilde{S} = diag\{\tilde{s}_1, \tilde{s}_2, ...\tilde{s}_n\}$. The input fictitious matrix B_f is defined as $B_f = \tilde{U}\tilde{S}^{\frac{1}{2}}$. $\tilde{P}_a$ mentioned in (3.41a) can also be written as solution of following discrete time Lyapunov equation:

$$A_{r_1}\tilde{P}_a A_{r_1}^T - \tilde{P}_a + B_f B_f^T = 0 \tag{3.42}$$

The transformation matrix T_2 is computed so that $T_2^T Q_h T_2 = T_2^{-1}\tilde{P}_a T_2^{-T} = \tilde{\Sigma} = diag\{\tilde{\sigma}_1, \tilde{\sigma}_2, ..., \tilde{\sigma}_{r_1}\}$ and $\tilde{\sigma}_1 \geq \tilde{\sigma}_2 \geq ... \geq \tilde{\sigma}_{r_1}$. After applying transformation we can express transformed system as $\{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\} = \{T_2^{-1}A_{r_1}T_2, T_2^{-1}B_{r_1}, C_{r_1}T_2, D\}$ which is then given by:

$$\left[\begin{array}{c|c} T_2^{-1}A_{r_1}T_2 & T_2^{-1}B_{r_1} \\ \hline C_{r_1}T_2 & D \end{array}\right] = \left[\begin{array}{cc|c} A_r & A_{r_{12}} & B_r \\ A_{r_{21}} & A_{r_{22}} & B_{r_2} \\ \hline C_{r_1} & C_{r_2} & D \end{array}\right]. \tag{3.43}$$

Then $H_r(z) = C_r(zI - A_r^{-1})B_r + D$ is the r^{th} order ROM such that $A_r \in \Re^{r\times r}, B_r \in \Re^{r\times m}$ and $C_r \in \Re^{m\times r}$. The following error bound holds for this stage of reduction if $rank\,[B_f\ B_{r_1}]$ =$rank\,[B_f]$ and $rank \begin{bmatrix} C_h \\ C_{r_1} \end{bmatrix}$ = $rank\,[C_h]$:

$$||W(z)(H_{r_1}(z) - H_r(z))||_\infty \leq 2||W(z)L_2||_\infty ||K_2||_\infty \sum_{k=r_1+1}^{n} \tilde{\sigma}_k$$

where

$$L_2 = C_{r_1}\tilde{V}diag\{\tilde{r}_1^{-\frac{1}{2}}, \tilde{r}_2^{-\frac{1}{2}}, ..., \tilde{r}_{\tilde{j}}^{-\frac{1}{2}}, 0, ..., 0\}$$

$$K_2 = diag\{\tilde{s}_1^{-\frac{1}{2}}, \tilde{s}_2^{-\frac{1}{2}}, ..., \tilde{s}_{\tilde{i}}^{-\frac{1}{2}}, 0, ..., 0\}\tilde{U}^T B_{r_1}$$

$\tilde{j} = rank\,[Y_2]$ and $\tilde{i} = rank\,[X_2]$.

The error bound for whole system is expressed as:

$$
\begin{aligned}
||W(z)(H(z)-H_r(z))V(z)||_\infty = ||W(z)(H(z)-H_{r_1}(z)+ \\
H_{r_1}(z)-H_r(z))V(z)||_\infty \leq ||W(z)||_\infty||(H(z)-H_{r_1}(z))V(z)||_\infty + \\
||W(z)(H_{r_1}(z)-H_r(z))||_\infty||V(z)||_\infty \leq 2||W(z)||_\infty||L_1||_\infty||K_1V(z)||_\infty \sum_{k=r_1+1}^{n}\tilde{\sigma}_k \\
+\,2||V(z)||_\infty||W(z)L_2||_\infty||K_2||_\infty \sum_{k_1=r+1}^{r_1}\tilde{\sigma}_{k_1}
\end{aligned}
$$

Theorem 4 *The ROM* $\{A_r, B_r, C_r, D\}$ *is guaranteed to be passive.*

Proof: The proof of above theorem is already mentioned in that of Proposition 1.

Algorithm 1: Step by Step procedure to calculate ROM $\{A_r, B_r, C_r, D\}$

1. Calculate Q and P_h using (3.28-3.30) and (3.6a-3.6c) respectively.
2. Compute Cholesky factor R_h of P_h i-e. $P_h = R_h^T R_h$.
3. Compute SVDs of $R_h Q R_h^T$ such that $R_h Q R_h^T = U_h \bar{\Sigma}^2 U_h^T$.
4. Compute T_1 as $T_1 = R_h^T U_h \bar{\Sigma}^{-\frac{1}{2}}$.
5. $H_{r_1}(z)$ is obtained using (3.32).
6. Compute $\tilde{P}_a$ and Q_h using (3.41) and (3.38-3.40) respectively.
7. Find the Cholesky factorization $\tilde{R}_a$ of $\tilde{P}_a$ i-e. $\tilde{P}_a = \tilde{R}_a^T \tilde{R}_a$.
8. Compute the SVDs of $\tilde{R}_a Q_h \tilde{R}_a^T$ such that $\tilde{R}_a Q_h \tilde{R}_a^T = \tilde{U}_a \bar{\Sigma}^2 \tilde{U}_a^T$.
9. Compute T_2 as $T_2 = \tilde{R}_a^T \tilde{U}_a \bar{\Sigma}^{-\frac{1}{2}}$.
10. The r^{th} order ROM is obtained using (3.43).

Remark 19 *The succession of stages could be exchanged i-e.* 1^{st} *being the output-weighted stage and the* 2^{nd} *one being the input-weighted stage or vice versa.*

Remark 20 *ROMs generated using this technique are not exclusive. Hence, the parameter* $r_1 (r < r_1 < n)$ *can be altered to accomplish better approximations.*

Numerical Examples

4.1 Introduction

In this chapter, different single and double sided frequency weighted numerical examples are presented to show the effectiveness of the schemes proposed in chapter 3. Simulations are performed using MATLAB 2018a on the computer system intel core i3, having 4 GB RAM and 2.20 GHz processor.

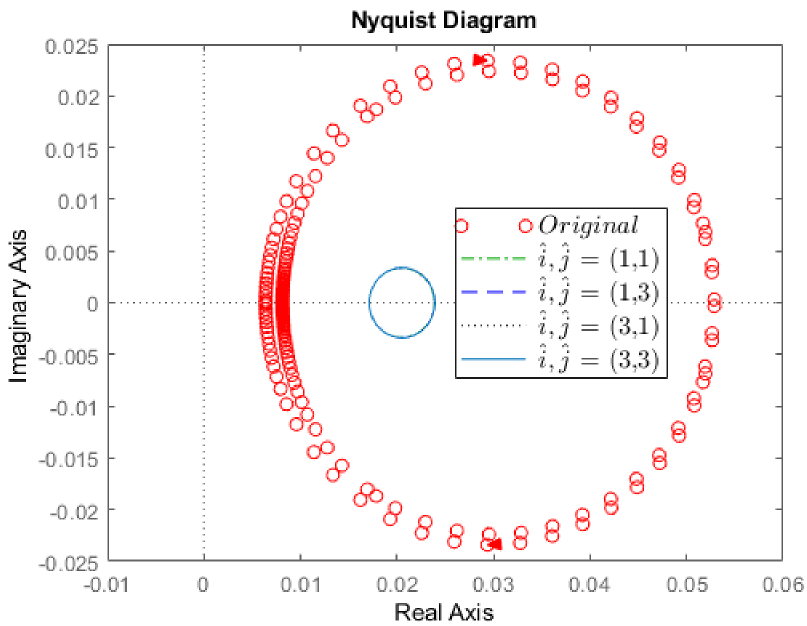

Figure 4.1: ROMs for $\{\hat{i},\hat{j}\}$ = (1,1),(1,3),(3,1),(3,3) of example 1

Example 1: Consider a 3^{rd} order passive DT original system given as:

$$H(z) = \frac{0.01713z^3 + 0.008051z^2 - 0.004368 + 0.0001233}{z^3 + 0.871z^2 + 0.6471z + 0.057}$$

with input weights

$$V(z) = \frac{0.1z}{z - 0.54}$$

and output weight

$$W(z) = \frac{0.1z^3 - 0.006z^2 + 8 \times 10^{-5}z}{z^3 - 0.19z^2 - 0.1794z - 0.005184}$$

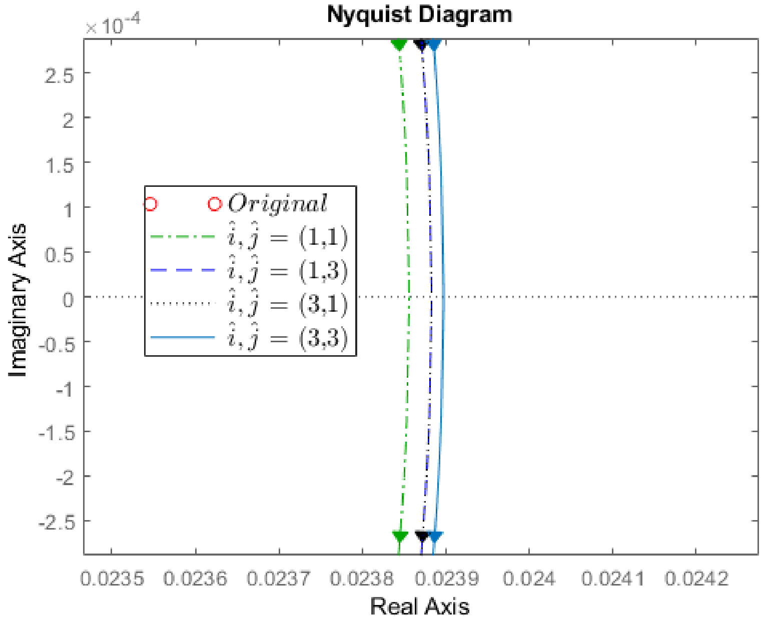

Figure 4.2: Zoomed portion for Fig. 4.1

reduced to 1^{st} order using different arrangements from Table 3.1. Fig. 4.1 shows Nyquist diagram of ROMs obtained using unweighted Gramians ($\{\hat{\imath}, \hat{\jmath}\} = \{1,1\}, \{1,3\}, \{3,1\}, \{3,3\}$). These Gramians were calculated using unweighted Lyapunov and Lur'e equations. It can be seen that Nyquist plot of these ROMs lie entirely in right half plane which shows their passivity. The Gramians obtained from unweighted Lyapunov equations ($\{\hat{\imath}, \hat{\jmath}\} = \{1,1\}$) don't guarantee passivity although in this example it generates passive ROM. However, passivity is guaranteed for ROMs obtained from unweighted Lur'e equations. Fig. 4.2 represents close-up view of Fig. 4.1. Fig. 4.3 shows Nyquist diagram of ROMs obtained using double sided frequency weighted Gramians ($\{\hat{\imath}, \hat{\jmath}\} = \{2,2\}, \{2,4\}, \{4,2\}, \{4,4\}$) and proposed technique. It can be seen that some part of Nyquist plot of these ROMs (except the one obtained from proposed technique) lie in left half plane which shows that they are non-passive whereas the Nyquist plot of proposed technique lies entirely in right half plane which shows its passivity. The arrangements $\{\hat{i}, \hat{j}\} = \{2,2\}, \{2,4\}, \{4,2\}, \{4,4\}$ generate non passive, however stable ROMs having poles at $z = -0.193105, -0.201718, -0.194259, -0.201715$. Note that in these scenarios generally stability is also not guaranteed.

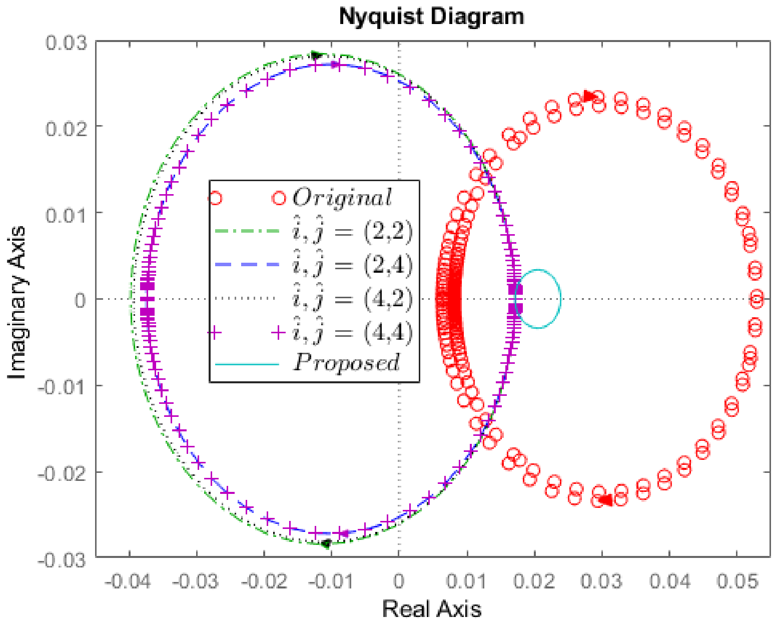

Figure 4.3: ROMs for $\{\hat{i}, \hat{j}\}$ = (2,2),(2,4),(4,2),(4,4) of example 1

Example 2: Consider a 100^{th} order RLC network with parameters $R_L = R_C = 1\Omega, L = 0.1H$ and $C = 0.01F$ discretized using ZOH and sampling time $T_s = 0.01s$ with input and output weightings as:

$$V(z) = W(z) = \frac{0.01713z}{z - 0.7754}$$

reduced to 5^{th} order using different arrangements from Table 3.1.

Fig.4.4 shows Nyquist diagram of ROMs obtained using unweighted Gramians ($\{\hat{i}, \hat{j}\} = \{1,1\}, \{1,3\}, \{3,1\}, \{3,3\}$). These Gramians were calculated using unweighted Lyapunov and Lur'e equations. It can be seen that Nyquist plot of these

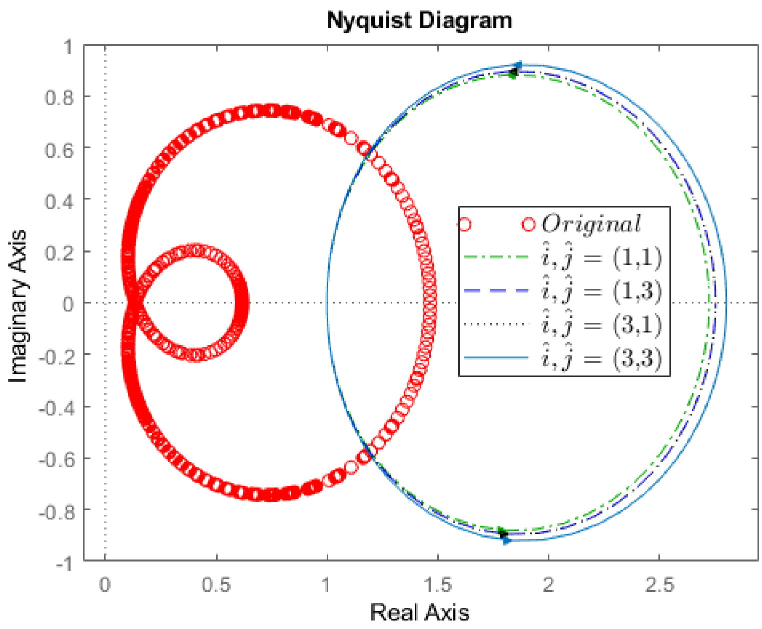

Figure 4.4: ROMs for $\{\hat{i},\hat{j}\}$ = (1,1),(1,3),(3,1),(3,3) of example 2

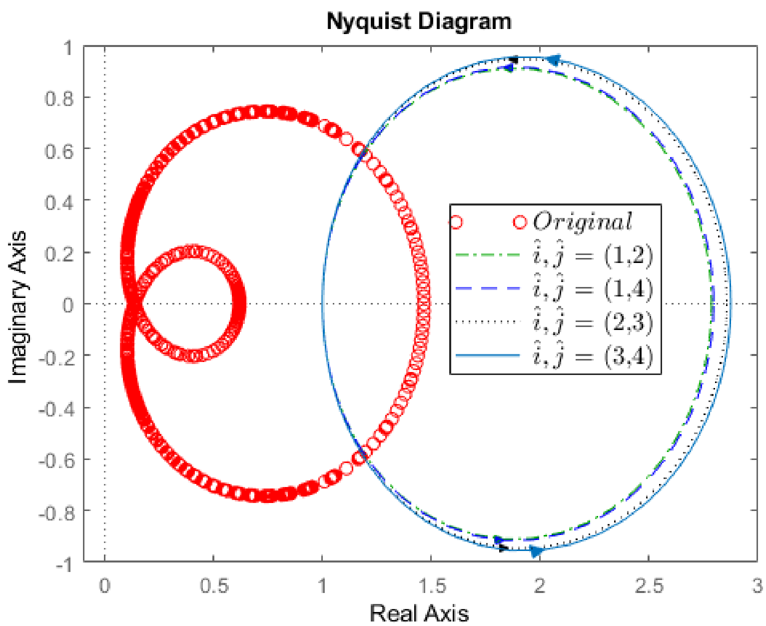

Figure 4.5: ROMs for $\{\hat{i},\hat{j}\}$ = (1,2),(1,4),(2,3),(3,4) of example 2

ROMs lie entirely in right half plane which shows their passivity. The Gramians obtained from unweighted Lyapunov equations ($\{\hat{i},\hat{j}\} = \{1,1\}$) don't guarantee passivity although in this example it generates passive ROM. However, passivity is guaranteed for ROMs obtained from unweighted Lur'e equations. Fig. 4.5

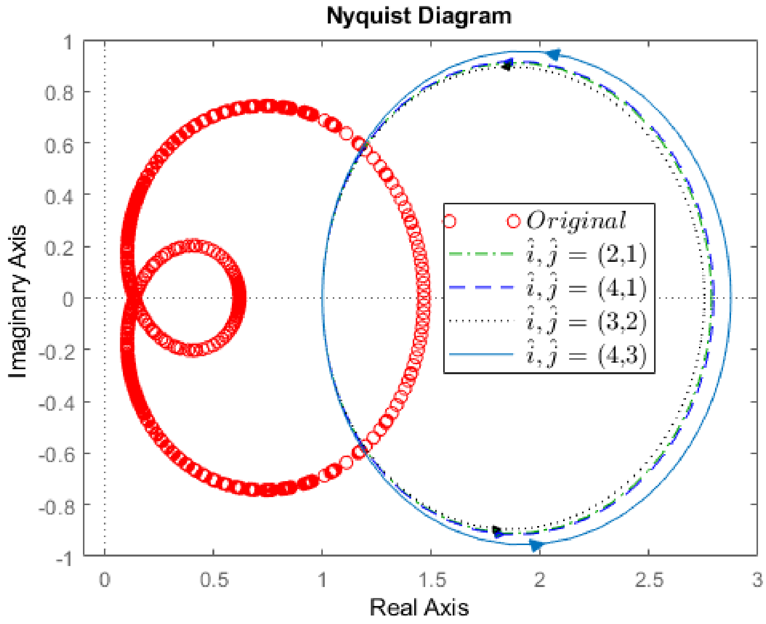

Figure 4.6: ROMs for $\{\hat{i},\hat{j}\}$ = (2,1),(4,1),(3,2),(4,3) of example 2

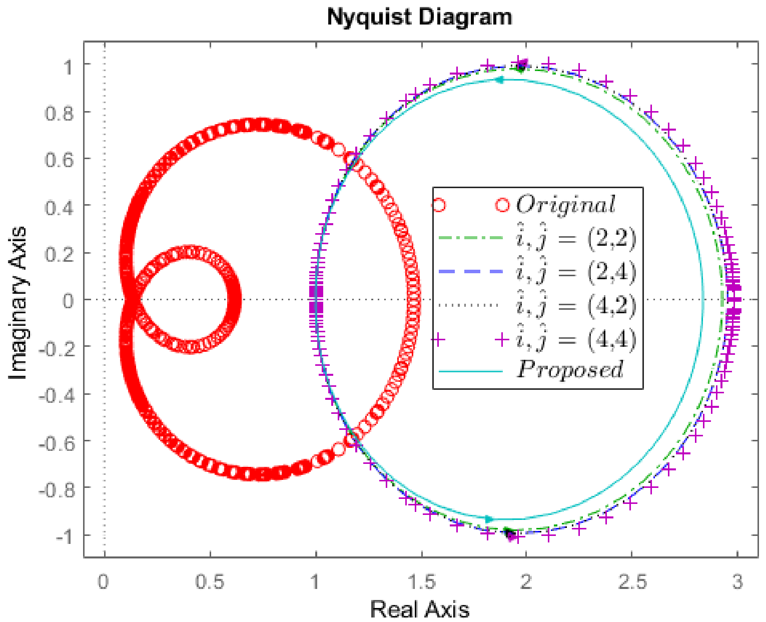

Figure 4.7: ROMs for $\{\hat{i},\hat{j}\}$ = (2,2),(2,4),(4,2),(4,4) of example 2

and Fig. 4.6 shows Nyquist diagram of ROMs obtained using single sided frequency weighted Gramians ($\{\hat{i},\hat{j}\} = \{1,2\},\{1,4\},\{2,3\},\{3,4\}$) and their duals ($\{\hat{i},\hat{j}\} = \{2,1\},\{4,1\},\{3,2\},\{4,3\}$) respectively. It can be seen that Nyquist plot of these ROMs also lie entirely in right half plane which shows their passivity. The

Gramians obtained from single sided frequency weighted Lyapunov equations don't guarantee passivity although in this example it generated passive ROM. However, passivity is guaranteed for the cases $\{\hat{i}, \hat{j}\} = \{4,1\}, \{3,2\}, \{4,3\}, \{4,1\}, \{3,2\}, \{4,3\}$. Fig. 4.7 shows Nyquist diagram of ROMs obtained using double sided frequency weighted Gramians ($\{\hat{i}, \hat{j}\} = \{2,2\}, \{2,4\}, \{4,2\}, \{4,4\}$) and proposed technique. It can be seen that some part of Nyquist plot of these ROMs lie entirely in right half plane which shows that they are passive. However, passivity in this case is not generally guaranteed (although in this example it generates passive ROMs). It can also be seen in Fig. 4.8 that approximation error between the ROM and original system is smallest in the proposed scheme than the arrangements (2,2),(2,4),(4,2),(4,4). Numerical values of error are also presented in Table 4.1. The advantage of proposed technique is that it guarantees passivity in case of double sided weightings and also generates less error.

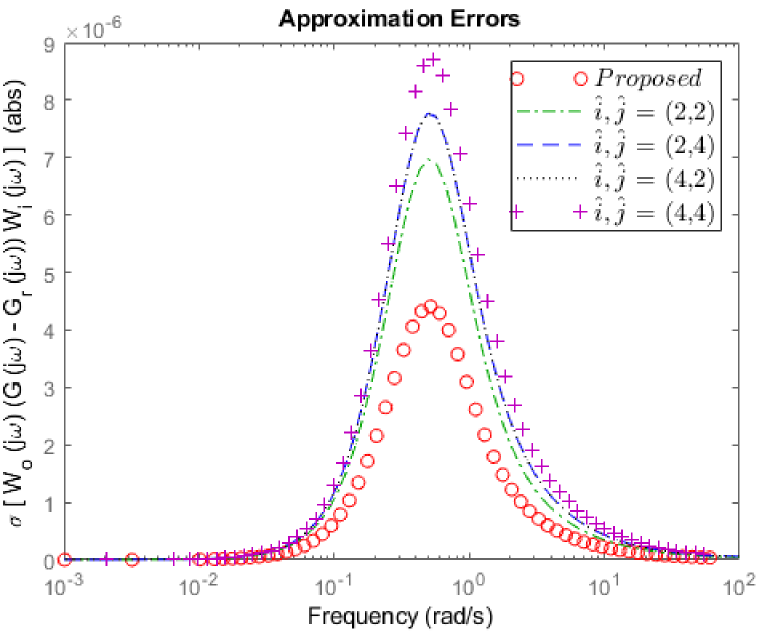

Figure 4.8: Error comparison plots for example 2

Table 4.1: Frequency Weighted Error Comparison of ROMs obtained in Example 2

$\hat{i},\hat{j}$ = (2,2)	6.97 $\times 10^{-6}$
$\hat{i},\hat{j}$ = (2,4)	7.78 $\times 10^{-6}$
$\hat{i},\hat{j}$ = (4,2)	7.78 $\times 10^{-6}$
$\hat{i},\hat{j}$ = (4,4)	8.71 $\times 10^{-6}$
Proposed	4.41 $\times 10^{-6}$

Example 3: Consider a 10^{th} order passive DT original system given as:

$$H(z) = (z^{10}-9.154z^9+37.86z^8-93.16z^7+151z^6-168.5z^5+131.1z^4-70.23z^3+24.78z^2-5.201z+0.493)\times(z^{10}-8.525z^9+32.72z^8-74.39z^7+110.9z^6-113.1z^5+79.86z^4-38.48z^3+12.08z^2-2.225z+0.1819)^{-1}$$

with input and output weights

$$V(z) = W(z) = \frac{0.1713z}{z-0.77544}$$

reduced to 1^{st} order using different arrangements from Table 3.1. Fig.4.9 shows Nyquist diagram of ROMs obtained using unweighted Gramians ($\{\hat{i},\hat{j}\} = \{1,1\},\{1,3\},\{3,1\},\{3,3\}$). These Gramians were calculated using unweighted Lyapunov and Lur'e equations. It can be seen that Nyquist plot of these ROMs lie entirely in right half plane which shows their passivity. The Gramians obtained from unweighted Lyapunov equations ($\{\hat{i},\hat{j}\} = \{1,1\}$) don't guarantee passivity although in this example it generates passive ROM. However, passivity is guaranteed for ROMs obtained from unweighted Lur'e equations. Fig. 4.10 and Fig. 4.11 shows Nyquist diagram of ROMs obtained using single sided frequency weighted Gramians ($\{\hat{i},\hat{j}\} = \{1,2\},\{1,4\},\{2,3\},\{3,4\}$) and their duals ($\{\hat{i},\hat{j}\} = \{2,1\},\{4,1\},\{3,2\},\{4,3\}$) respectively. It can be seen that Nyquist plot of these ROMs also lie entirely in right half plane which shows their passivity. The Gramians obtained from single sided frequency weighted Lyapunov equations don't

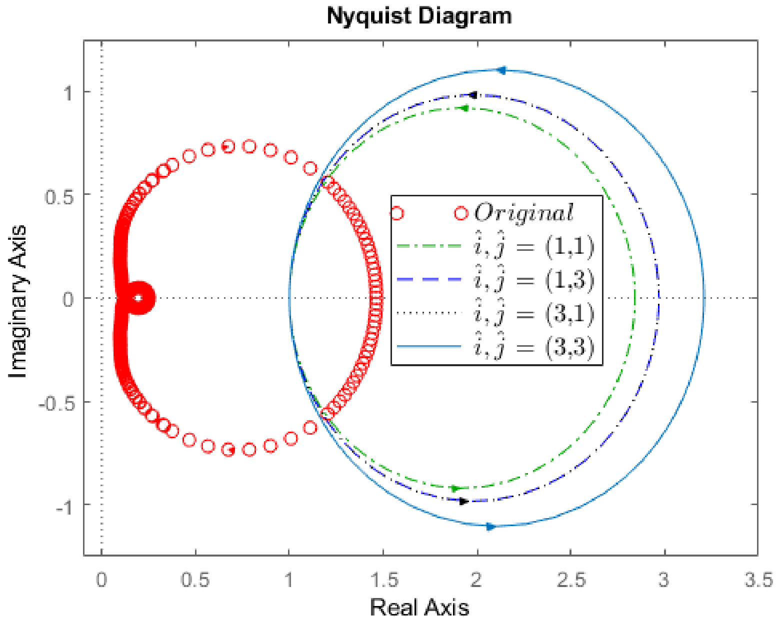

Figure 4.9: ROMs for $\{\hat{i},\hat{j}\} =$ (1,1),(1,3),(3,1),(3,3) of example 3

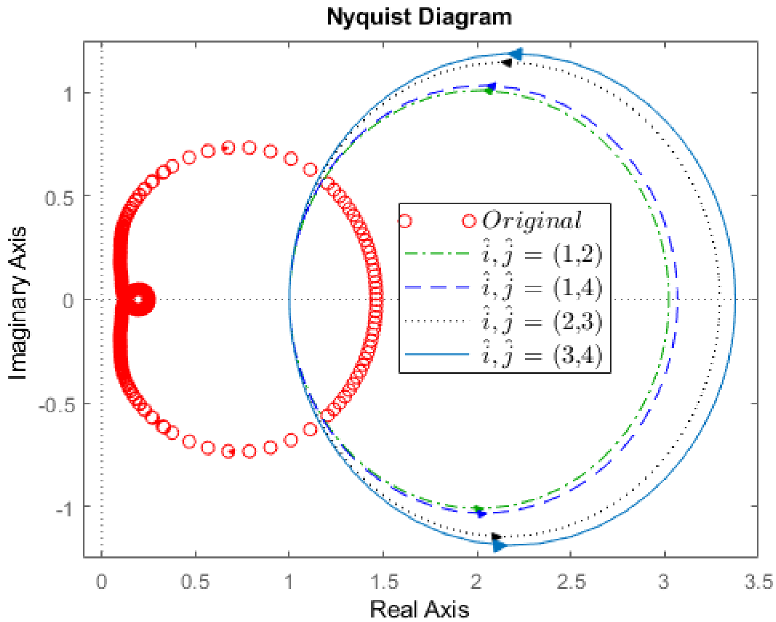

Figure 4.10: ROMs for $\{\hat{i},\hat{j}\} =$ (1,2),(1,4),(2,3),(3,4) of example 3

guarantee passivity although in this example it generated passive ROM. However, passivity is guaranteed for the cases $\{\hat{i},\hat{j}\} = \{4,1\},\{3,2\},\{4,3\},\{4,1\},\{3,2\},\{4,3\}$. Fig. 4.12 shows Nyquist diagram of ROMs obtained using double sided frequency weighted Gramians ($\{\hat{i},\hat{j}\} = \{2,2\},\{2,4\},\{4,2\},\{4,4\}$) and proposed technique. It

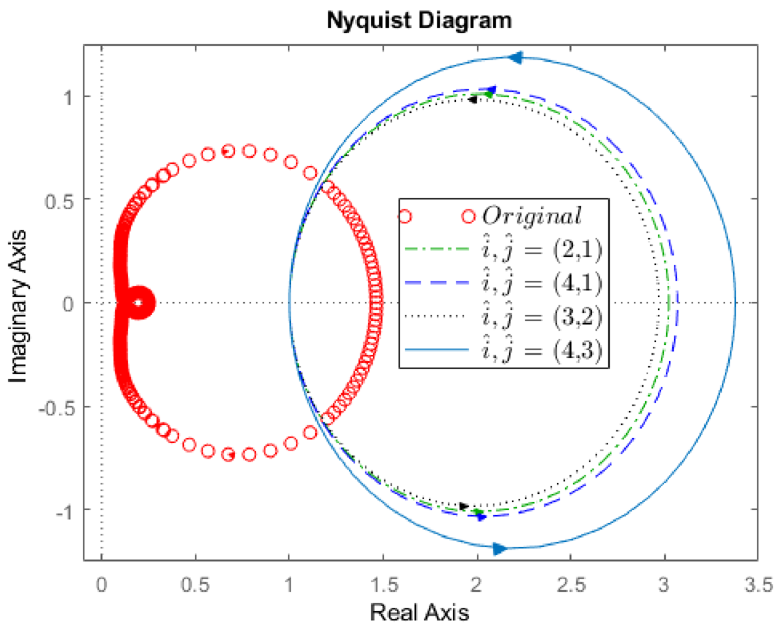

Figure 4.11: ROMs for $\{\hat{i},\hat{j}\} = (2,1),(4,1),(3,2),(4,3)$ of example 3

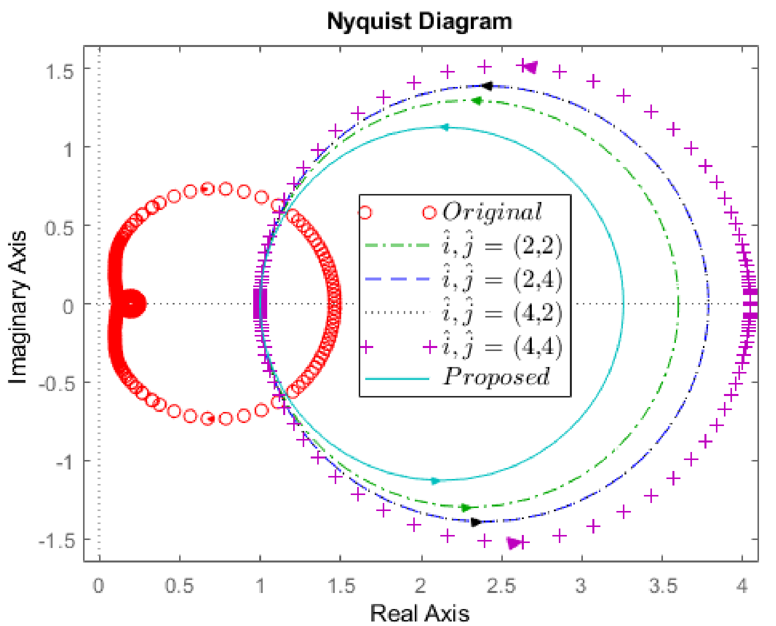

Figure 4.12: ROMs for $\{\hat{i},\hat{j}\} = (2,2),(2,4),(4,2),(4,4)$ of example 3

can be seen that some part of Nyquist plot of these ROMs lie entirely in right half plane which shows that they are passive. However, passivity in this case is not generally guaranteed (although in this example it generates passive ROMs). It can also be seen in Fig. 4.13 that approximation error between the ROM and original system is smallest

Table 4.2: Frequency Weighted Error Comparison of ROMs obtained in Example 3

$\hat{i}, \hat{j}$ = (2,2)	2.07×10^{-5}
$\hat{i}, \hat{j}$ = (2,4)	2.41×10^{-5}
$\hat{i}, \hat{j}$ = (4,2)	2.41×10^{-5}
$\hat{i}, \hat{j}$ = (4,4)	2.85×10^{-5}
Proposed	1.36×10^{-5}

in the proposed scheme than the arrangements (2,2),(2,4),(4,2),(4,4). Numerical values of error are also presented in Table 4.2. The advantage of proposed technique is that it guarantees passivity in case of double sided weightings and also generates less error.

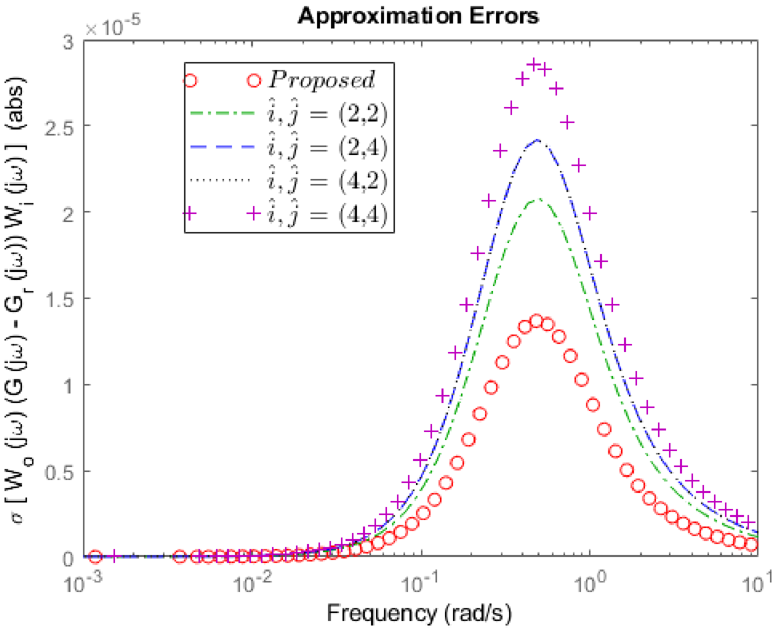

Figure 4.13: Error comparison plots for example 3

Example 4: Consider a 8^{th} order passive DT original system represented with following state space:

$$A = \begin{bmatrix} 0.9587 & 0.0000 & 0.0000 & 0.0108 & -0.0001 & -0.0000 & 0.0155 & -0.0212 \\ 0.0000 & 0.9558 & 0.0100 & 0.0108 & 0.0234 & -0.0212 & 0.0000 & 0.0001 \\ 0.0000 & 0.0100 & 0.7686 & 0.0000 & 0.0001 & 0.0209 & 0.0000 & 0.0000 \\ 0.0108 & 0.0108 & 0.0000 & 0.9558 & -0.0212 & -0.0001 & 0.0001 & 0.0234 \\ 0.0032 & -0.9345 & -0.0034 & 0.8490 & 0.8919 & 0.0105 & 0.0000 & 0.0105 \\ 0.0000 & 0.8492 & -0.8376 & 0.0032 & 0.0105 & 0.8927 & 0.0000 & 0.0000 \\ -0.6186 & -0.0000 & -0.0000 & -0.0029 & 0.0000 & 0.0000 & 0.3614 & 0.0080 \\ 0.8496 & -0.0036 & -0.0000 & -0.9345 & 0.0105 & 0.0000 & 0.0080 & 0.8919 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0091 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.6296 \\ 0.0029 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, D = 1$$

with input and output weights

$$V(z) = W(z) = \frac{z^2 + 0.2z + 0.324}{z^2 + 0.354z + 0.22}$$

reduced to 1^{st} order using different arrangements from Table 3.1. Fig.4.14 shows Nyquist diagram of ROMs obtained using unweighted Gramians ($\{\hat{i}, \hat{j}\} = \{1,1\}, \{1,3\}, \{3,1\}, \{3,3\}$). These Gramians were calculated using unweighted Lyapunov and Lur'e equations. It can be seen that Nyquist plot of these ROMs lie entirely in right half plane which shows their passivity. The Gramians obtained from unweighted Lyapunov equations ($\{\hat{i}, \hat{j}\} = \{1,1\}$) don't guarantee passivity although in this example it generates passive ROM. However, passivity is guaranteed for ROMs obtained from unweighted Lur'e equations.

Fig. 4.15 and Fig. 4.16 shows Nyquist diagram of ROMs obtained using single sided

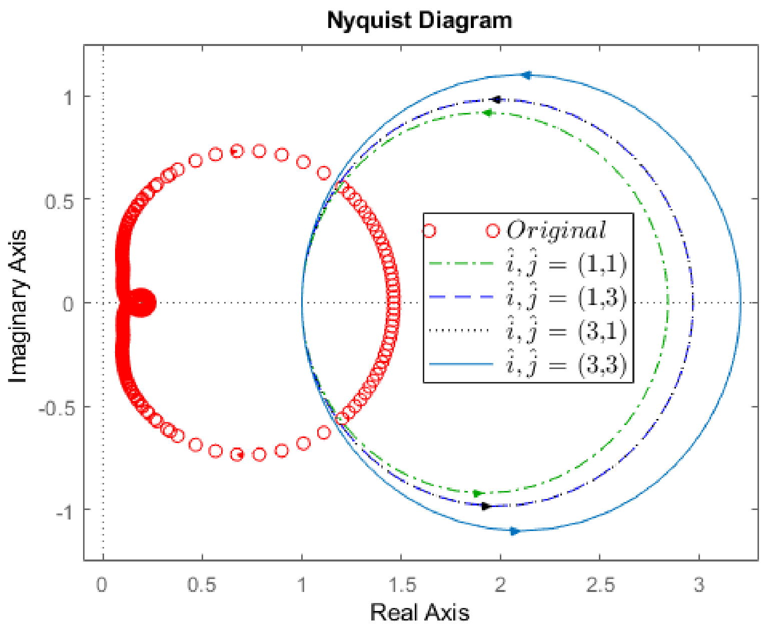

Figure 4.14: ROMs for $\{\hat{i},\hat{j}\}$ = (1,1),(1,3),(3,1),(3,3) of example 4

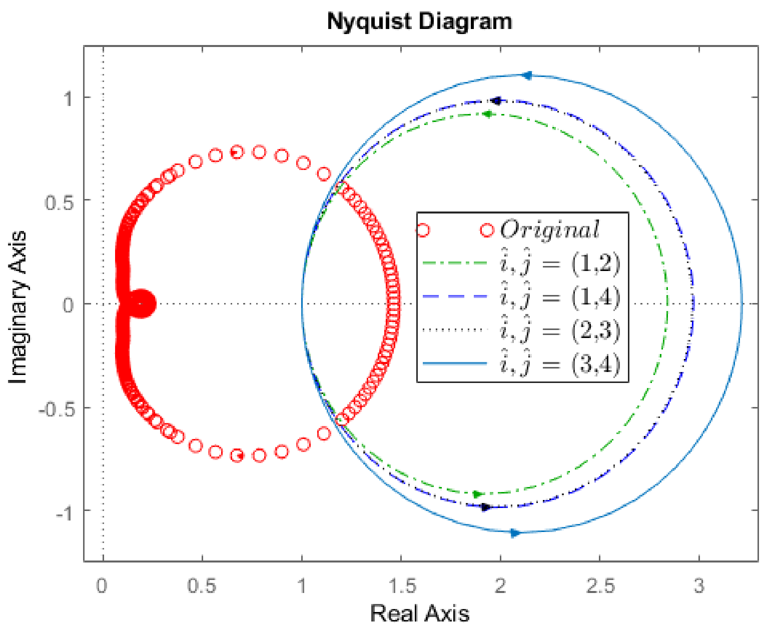

Figure 4.15: ROMs for $\{\hat{i},\hat{j}\}$ = (1,2),(1,4),(2,3),(3,4) of example 4

frequency weighted Gramians ($\{\hat{i},\hat{j}\} = \{1,2\},\{1,4\},\{2,3\},\{3,4\}$) and their duals ($\{\hat{i},\hat{j}\} = \{2,1\},\{4,1\},\{3,2\},\{4,3\}$) respectively. It can be seen that Nyquist plot of these ROMs also lie entirely in right half plane which shows their passivity. The Gramians obtained from single sided frequency weighted Lyapunov equations don't

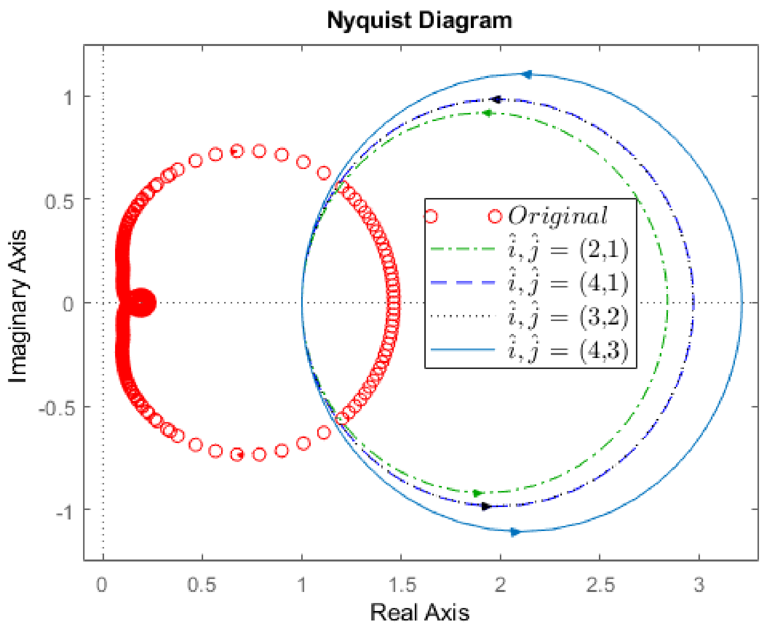

Figure 4.16: ROMs for $\{\hat{i}, \hat{j}\}$ = (2,1),(4,1),(3,2),(4,3) of example 4

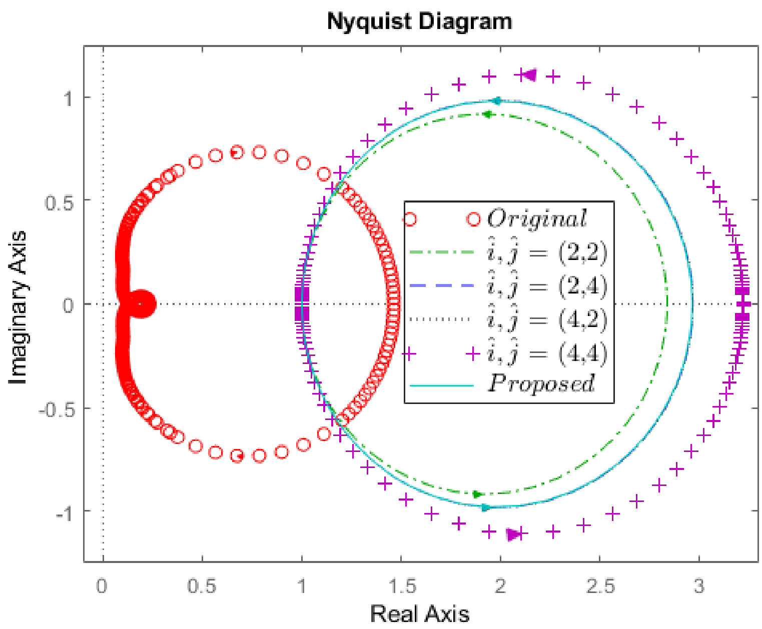

Figure 4.17: ROMs for $\{\hat{i}, \hat{j}\}$ = (2,2),(2,4),(4,2),(4,4) of example 4

guarantee passivity although in this example it generated passive ROM. However, passivity is guaranteed for the cases $\{\hat{i}, \hat{j}\} = \{4, 1\}, \{3, 2\}, \{4, 3\}, \{4, 1\}, \{3, 2\}, \{4, 3\}$.

Fig. 4.17 shows Nyquist diagram of ROMs obtained using double sided frequency weighted Gramians ($\{\hat{i}, \hat{j}\} = \{2, 2\}, \{2, 4\}, \{4, 2\}, \{4, 4\}$) and proposed technique. It

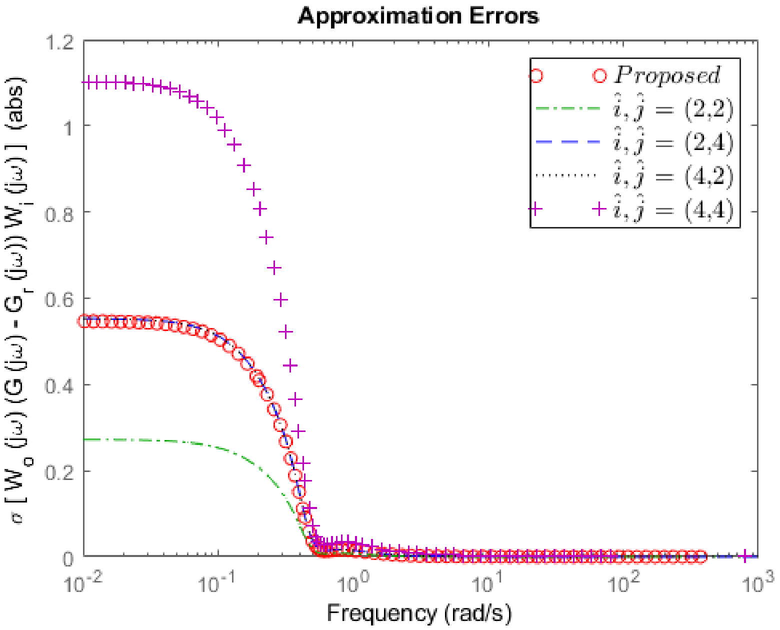

Figure 4.18: Error comparison plots for example 4

can be seen that some part of Nyquist plot of these ROMs lie entirely in right half plane which shows that they are passive. However, passivity in this case is not generally guaranteed (although in this example it generates passive ROMs). The approximation error can be seen in Fig. 4.18 between the ROM and original system. Numerical values of error are also presented in Table 4.3. We can see that arrangement (2,2) generates less error in this example but it doesn't guarantees passivity in case of double sided frequency weightings. The advantage of proposed technique is that it guarantees passivity in case of double sided weightings and also generates comparatively less error.

Table 4.3: Frequency Weighted Error Comparison of ROMs obtained in Example 4

$\hat{i}, \hat{j}$ = (2,2)	0.2725
$\hat{i}, \hat{j}$ = (2,4)	0.5536
$\hat{i}, \hat{j}$ = (4,2)	0.5536
$\hat{i}, \hat{j}$ = (4,4)	1.1041
Proposed	0.5471

4.2 Conclusion

In this chapter several numerical examples have been presented which show the effectiveness of proposed schemes. The passivity is always preserved in case of single side frequency weightings whereas it is not guaranteed in case of double sided weightings. Therefore scheme was proposed to preserve passivity in case of double sided weightings for discrete time systems. Approximation errors of the proposed technique is less than that obtained from cases $\{i, j\} = \{2, 2\}, \{2, 4\}, \{4, 2\}, \{4, 4\}$.

Chapter 5

Conclusion and future work

5.1 Overview of the Thesis

This thesis has explored the problem of passivity preserving for frequency weighted MOR for DT systems. Both stability and passivity preserving techniques were studied. In chapter 2, a brief analysis of existing MOR techniques was presented which takes account of both stability and passivity preserving problems for frequency weighted and unweighted cases. In chapter 3, a family of technique were proposed to preserve passivity in DT systems. Some techniques preserve only stability while other techniques do not preserve stability as well as passivity. All the proposed techniques were critically examined and several remarks were presented about their behavior whether they preserve or do not preserve basic properties of a system like stability and passivity etc. In chapter 4, numerical examples with double sided frequency weights were presented to show the usefulness of the proposed schemes discussed in chapter 3. Simulation results and mathematical equations/derivations show that the proposed techniques serve the purpose.

5.2 Conclusion

In this thesis, a family of MOR techniques based on passivity preserving for single and double sided frequency weights are presented. Simulation results show that the proposed techniques preserve passivity of a ROM in the desired frequency range in case of single sided frequency weight and some of the proposed techniques preserve passivity of a ROM in the desired frequency range in case of double sided frequency weights.

5.3 Future Work

In this section, we suggest/recommend that this research can be further enhanced to a level where one can desire. For future directions it is recommended: spcaing

* The optimization technique can be applied to error bound for better approximations.

* Lyapunov stability criteria and Lur'e/ARE passivity criteria are not necessary to yield stable and passive ROM's, respectively. This area is also open to yield efficient stable and passive ROM's in case of double sided frequency weights.

* The computational cost/memory of the proposed techniques in terms of Lyapunov equations and ARE's/Lure equations can be improved by using new efficient algorithms.

* As the proposed techniques are realization dependent and it is unknown that which new realization produces less approximation error, so it is also an open question and needs further research.

* Selection of frequency weights for least approximation error need more investigation because different frequency weights yield different approximation errors/results.

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