IMPROVED TIME LIMITED MODEL ORDER REDUCTION TECHNIQUES

By

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ABSTRACT

Model order reduction (MOR) is a technique developed in the area of control system theory, which reduces the complexity of higher order systems by reducing the order of the system while retaining the key features of original system. These models are represented by partial differential equations, ordinary differential equations. MOR approximates higher-order original models by relatively lower-order models to give simplicity in design, modeling and simulation for huge complicated systems. The analysis of large scale models is difficult or even sometimes impossible to perform due to different constraints like storage, cost and computation. Therefore MOR techniques are developed. The Balanced truncation(BT) [1] is one of the most frequently used MOR technique because reduced order models (ROMs) obtained using this technique are not only stable but also have quantifiable error bounds. In BT [1] MOR technique lower energy states are truncated and higher energy states are retained to get ROM having similar characteristics as original system. Considerable amount of research has been done on different features of MOR. Existing techniques have the drawbacks of lacking properties like stability, passivity and large approximation error produced in ROMs etc. Ideally BT [1] technique approximates the higher order system by relatively lower order system having low approximation error for entire time interval. Though, in some applications approximation error is required to be small for specific time interval rather than for the entire time range. Therefore time limited MOR techniques are developed in which controllability and observability Gramians are defined over finite time interval. The goal is to achieve a stable ROM having the same response characteristics as the original system and low approximation error.

This thesis includes Time Limited Gramians based model order reduction (TLMOR) techniques for standard continuous time systems . The proposed techniques produce less approximation error as contrast to existing techniques. Numerical examples are also illustrated to exhibit the compatibility and effectiveness of the proposed techniques to the existing ones. Some of practical applications of MOR are

- Fabrication industries
- Missiles analysis and launching
- Industrial real time applications
- Radio frequency micro electro-mechanical systems (RF MEMS)

DEDICATION

I dedicate this thesis to

MY PARENTS, FRIENDS AND TEACHERS

who have been a great source of inspiration and support.

ACKNOWLEDGEMENTS

Firstly, I would like to thank Allah Almighty for giving me the strength, knowledge and opportunity to undertake this research. Without His blessings, this achievement would not have been possible. I would like to thank my supervisor Assistant Professor Dr. Muhammad Imran for his guidance through this process. I am especially grateful to my parents, who supported me in every step of this journey.

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ACRONYMS

Balanced Truncation	BT
Model Order Reduction	MOR
Reduced Order Models	ROMs
Singular Value Decomposition	SVD
Linear Time Invariant	LTI
Frequency Weighted Model Order Reduction	FWMOR
Frequency Limited Model Order Reduction	FLMOR
Time Limited Model Order Reduction	TLMOR
Gawronski and Juang's Technique	GJ
Gugercin and Antoulass technique	GA
Ghafoor ad Sreeram's Technique	GS
Imran and Ghafoor's Technique	IG
Imran's TLMOR Technique	MI

Chapter 1

Introduction

1.1 Overveiw of Model Order Reduction

Each physical system can be represented as a mathematical model. Manifestation of physical systems produce complex higher order mathematical models. These higher order models produce complex differential condition that are difficult to analyze, design , mimic and store, and also take much memory for capacity. To address these issues, a strategy is required to bring down the computational cost by reducing the order of the system that retains the basic parameters of the original system like input output behaviour, stability and lower approximation error of original systems. The process of reducing the order of the system to get a lower order mathematical model is known as Model order reduction (MOR) which contributes an important part in control system theory analysis and design.

1.2 A brief overview of MOR Techniques

In this section, different stability preserving MOR techniques (BT [1], frequency weighted model order reduction (FWMOR), frequency limited model order reduction (FLMOR) and time limited model order reduction (TLMOR)) are discussed.

1.2.1 Balanced truncation

BT [1] is one of the popular techniques used in MOR. The idea of BT [1] was given by Moore. In BT [1] the original system is approximated as a ROM over the entire frequency/time interval where the original system realization is transformed into an internally balanced realization using a contragradient transformation T and then ROM is achieved by diminishing the least controllable and least observable states of the balanced realization.

1.2.2 Frequency weighted model order reduction

Most of the physical systems work on a limited frequency interval and the response of the systems outside this range is not very important. This is true for the case, when ROMs are used in feedback control design [2, 3]. This leads the concept of using frequency weights in

MOR procedure. Enns [2] extended the BT [1] technique by introducing frequency weights on the input and output side of the system. Enns [2] technique produces stable ROM when weights are applied only on one side. But when frequency weights are applied on both input and output side of the system the ROM may become unstable [4]. To overcome this issue of instability many authors presented different techniques in literature [5]- [6]. To conquer Enns method [2] disadvantages, Lin and Chiu [7] has proposed an alternate procedure that gives the assurance of stability of ROM when two sided weightings are available. In any case, their strategy has a confinement that can work just when entirely appropriate weighting capacity is utilized as a part of and no pole, zero cancelation happens while shaping the augumented system. These shortcomings of Lin and Chui [7] method were later adjusted by Sreeram et al [8] and Varga and Anderson (VA) [6], where [8] summed up [7] to incorporate weights, while [6] holds the dependability of the system not withstanding when shaft zero cancelation happen. VA [6] produces an indistinguishable outcomes from Enns [2] particularly in controller reduction applications. So far controller reduction issue, if Enns system [2] produces unstable ROMs, so does by VA [6] method.

Wang et al's method [9] has likewise solve the stability issue of Enns [2], which not just give stable ROMs within the presence of two sided weightings additionally yield error bound. The approximation error of Wang et al method [9] was later enhanced by VA [6] as commented by Sreeram [10]. This system and its adjustment by VA [6] are realization independent. This implies for a similar unique system , diverse models can be attained from various realizations.

1.2.3 Frequency limited model order reduction

FLMOR is a technique in which frequency weights are not used but Gramians are defined over a limited frequency interval. Gawronski and Juang (GJ) [11] gave the idea of frequency limited Gramians in MOR. GJ [11] technique produces less approximation error but does not ensure the stability of ROM. In addition , there are no error bounds. Many authors presented different techniques to conquer the issue of instability including Gugercin and Antoulas (GA) [12], Ghafoor and Sreeram (GS) [4], Imran and Ghafoor (IG) [13], Imran, Ghafoor and Imran (IGI) [14], Shafiq, Ghafoor and Imran [15] and Zulfiqar, Imran and Ghafoor [16]. GA [12] modified the GJ [11] technique by making the symmetric matarices positive definite. GA [12] technique ensures the stablity of the ROM. GS [4] exhibited another alteration to GJ [11] system to give stable ROMs. It also carries frequency response error bound to satisfy rank conditions. IG [13] presented a technique which produces stable ROM also give frequency response error bound. IGI [14] presented multiple techniques to conquer the issue of stability in GJ [11]. IGI [14] techniques provides less approximation error as compared to existing techniques and also produce frequency response error bound.

1.2.4 Time limited model order reduction

GJ [11] also presented a technique in which Gramians are defined over a limited time interval known as time limited model order reduction technique (TLMOR). GJ [11] technique of TLMOR likewise does not guarantee to produce stable ROMs and does not have frequency response error bound. (GA) [12] modified the GJ [11] TLMOR technique to conquer the issue of instability and presented a technique of TLMOR which yields stable ROMs but this technique give large approximation error. Muhammad Imran (MI) [17] presented two techniques of TLMOR. Both techniques produces stable ROM. Model reduction of large scale descriptor systems using TLMOR technique appear in [18]. Kumar, Jazlan and Sreeram [19] presented a TLMOR technique to rectify the problem of instability of ROM. In this thesis two TLMOR techniques are presented. Both techniques ensure the stability of ROM and also give least approximation error as compared to the existing TLMOR techniques. Numerical examples are also illustrated to show the effectiveness of the proposed techniques. TLMOR is widely used in electrical circuit recreation and small scale electro-mechanical framework, parameter evaluation with a particular attention to real-time computing in biomedical engineering and computational physics, the study of high-dimensional problems in state space, physical space. MOR are widely used in prediction of real world systems like climate or the human cardiovascular system where large complex mathematical models are used.

1.3 Problem Summary

Existing TLMOR techniques may yeild unstable ROMs, and yield more estimation error.

1.4 Contributions

The summary of the thesis is stated as,

- In this thesis two TLMOR techniques are proposed which guarantee the stability of ROM.
- Proposed techniques deliver less approximation error when contrasted with existing TLMOR techniques.

1.5 Thesis Outline

This thesis is separated into five parts:

- Chapter 1: In this chapter, a brief introduction of different MOR techniques (BT, FWMOR, FLMOR, TLMOR) is illustrated.
- Chapter 2: This chapter incorporates all the existing FWMOR, FLMOR and TLMOR methods.
- Chapter 3: This chapter incorporates the proposed stability preserving TLMOR methods.
- Chapter 4: In this chapter numerical examples and their results are discussed to show the efficacy of the proposed TLMOR techniques.
- Chapter 5: Future work and Conclusion are presented in this section

Chapter 2

Existing Techniques

In this chapter existing techniques of MOR are discussed. BT [1] is one of the most popular techniques used for MOR that preserves stability in ROMs. Enns [2] extended BT [1] and introduced frequency weights in MOR. But Enns [2] technique produce unstable ROM in case of double sided weighting [20]. Different authors proposed different techniques to conquer this instability issue. GJ [11] proposed a MOR technique in which frequency weights are not defined but he approximated the original system in a limited frequency interval by producing less approximation error. But GJ [11] technique do not ensures the stability of ROMS [12]. Many authors presented different techniques to overcome this issue of instability. GJ [11] also presented an idea of TLMOR. GJ [11] technique of TLMOR likewise does not ensures the stability of ROM. GA [12] extended the GJ [11] technique of TLMOR to ensures the stability of ROM but ROM obtained using GA [12] leads to large approximation error. MI [17] presented two techniques of TLMOR. Both techniques produces stable ROM but leads to large error.

2.1 Preliminaries

Let a n^{th} order stable system $G(s) = C(sI - A)^{-1}B + D$ since $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times n}$, $C \in \mathcal{R}^{p\times n}$, $D \in \mathcal{R}^{p\times m}$ where inputs and outputs are defined as m and p respectively. A MOR problem is to find

$$G_{kk}(s) \quad = \quad C_1(sI - A_{11})^{-1}B_1 + D_1 \tag{2.1}$$

which approximates the original system (in the time range $[t_1, t_2]$,$t_2 > t_1$) in that case $A_{11} \in \mathcal{R}^{k\times k}$, $B_1 \in \mathcal{R}^{k\times m}$, $C_1 \in \mathcal{R}^{p\times k}$, $D_1 \in \mathcal{R}^{p\times m}$, $k < n$. Let P_i and Q_o are controlability and observibility Gramians respectively, satisfying following Lyapunov equations:

$$P_i \quad = \quad \int_{t_1}^{t_2} e^{A\tau}BB^Te^{A^T\tau}d\tau \tag{2.2}$$

$$Q_o \quad = \quad \int_{t_1}^{t_2} e^{A^T\tau}C^TCe^{A\tau}d\tau \tag{2.3}$$

The controlability and observibility Gramians P_i and Q_o are the solution of the following Lyapunov equations.

$$AP_i + P_iA^T + BB^T = 0 \tag{2.4}$$

$$A^TQ_o + Q_oA + C^TC = 0 \tag{2.5}$$

2.2 Balanced Truncation

BT [1] is the most utilized technique for MOR. In BT [1] the realization of original system is transformed into a internally balanced realization using a contragradient transformation matrix T. ROM is obtained by segregating the least controllable and observable states of system realization. This makes the estimation error smaller in utilizing BT [1] system, which is viewed as a decent execution of ROMs. Other than BT [1], other such plans, for example, Hankel ideal estimate [21], Pade apprximation [22] , Krylov technique [23] and so on are valuable for limiting MOR disadvantages. BT [1] is a good option for higher frequencies as it produces good results. Let the original n^{th} order system is

$$G(s) = C(sI - A)^{-1}B + D \tag{2.6}$$

where $\{A, B, C, D\}$ is its n^{th} order minimal realization. Let P_i and Q_o be the controllability and the observability Gramians satisfying these Lyapunov equations:

$$AP_i + P_iA^T + BB^T = 0 \tag{2.7}$$

$$A^TQ_o + Q_oA + C^TC = 0 \tag{2.8}$$

In order to find a stable ROM of order k, where $k < n$ the system realization is transformed into a balanced realization using the contragradient transformation T_b and then is truncated by deleting the least controllable and least observable states. Let contragradient transformation T_b is obtained such that

$$T_b^TQT_b = T_b^{-1}PT_b^{-T} = \begin{bmatrix} \sum_{b_1} & 0 \\ 0 & \sum_{b_2} \end{bmatrix}$$

where $\sum_{b_1} = diag\{\sigma_1, \sigma_2, ..., \sigma_k\}$, $\sum_{b_2} = diag\{\sigma_{k+1}, ..., \sigma_n\}$, $\sigma_i \geq \sigma_{i+1}$, $i = 1, 2, ..., n-1$, $\sigma_k \geq \sigma_{k+1}$. Applying the contragradient transformation T_b to the original system

$$\left[\begin{array}{c|c} A_b & B_b \\ \hline C_b & D_b \end{array}\right] = \left[\begin{array}{c|c} T_b^{-1}AT_b & T_b^{-1}B \\ \hline CT_b & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \quad (2.9)$$

The ROM is obtained by truncating the balanced realization

$$G_{kk}(s) = C_1(sI - A_{11})^{-1}B_1 + D_1$$

where $\{A_{11}, B_1, C_1, D_1\}$ is the k^{th} $(k < n)$ order minimal realization.

2.3 Frequency Weighted Model Order Reduction

Enns [2] extended BT [1] method by introducing frequency weights in MOR. Ideally BT [1] technique approximates the higher order system by relatively lower order system having low approximation error for entire fequency/time interval. Though, in some applications approximation error is required to be small for specific frequency/time interval rather than for the entire frequency/time range. This motivates the use of frequency weights in MOR. Enns [2] used frequency weights on input side, output side or both sides. When weighting function is used only on one side either on input or output side, stability of ROM is guaranteed. But when both input and output weighting functions are used this technique may yield unstable ROM. Given the original full order stable system

$$G(s) = C(sI - A)^{-1}B + D \quad (2.10)$$

The stable input weighting system $V_{in}(s)$ and stable output weighting system $W_{ou}(s)$ are respectively

$$V_{in}(s) = C_{in}(sI - A_{in})^{-1}B_{in} + D_{in} \quad (2.11)$$

$$W_{ou}(s) = C_{ou}(sI - A_{ou})^{-1}B_{ou} + D_{ou} \quad (2.12)$$

where $\{A, B, C, D\}$, $\{A_{in}, B_{in}, C_{in}, D_{in}\}$ and $\{A_{ou}, B_{ou}, C_{ou}, D_{ou}\}$ are the n^{th}, p^{th}, q^{th} order minimal realization of original system, input weighting system and output weighting system respectively. The objective of FWMOR is to find a lower order stable system

$$G_{kk}(s) = C_1(sI - A_{11})^{-1}B_1 + D_1 \tag{2.13}$$

where $\{A_{11}, B_1, C_1, D_1\}$ is the k^{th} $(k < n)$ order minimal realization. The input augmented system and output augmented systems are given by

$$G(s)V_{in}(s) = C_{ai}(sI - A_{ai})^{-1}B_{ai} + D_{ai} \tag{2.14}$$

$$W_{ou}(s)G(s) = C_{ao}(sI - A_{ao})^{-1}B_{ao} + D_{ao} \tag{2.15}$$

where

$$\{A_{ai}, B_{ai}, C_{ai}, D_{ai}\} = \left\{ \begin{bmatrix} A & BC_{in} \\ 0 & A_{in} \end{bmatrix}, \begin{bmatrix} BD_{in} \\ B_{in} \end{bmatrix}, \begin{bmatrix} C & DC_{in} \end{bmatrix}, DD_{in} \right\} \tag{2.16}$$

$$\{A_{ao}, B_{ao}, C_{ao}, D_{ao}\} = \left\{ \begin{bmatrix} A_{ao} & B_{ao}C \\ 0 & A \end{bmatrix}, \begin{bmatrix} B_{ao}D \\ B \end{bmatrix}, \begin{bmatrix} C_{ao} & D_{ao}C \end{bmatrix}, D_{ao}D \right\} \tag{2.17}$$

Let

$$P_{ai} = \begin{bmatrix} P_e & P_{12} \\ P_{12}^T & P_{in} \end{bmatrix}$$

$$Q_{ao} = \begin{bmatrix} Q_{ou} & Q_{12}^T \\ Q_{12} & Q_e \end{bmatrix}$$

be the Gramians satisfying the following Lyapunov equations.

$$A_{ai}P_{ai} + P_{ai}A_{ai}^T + B_{ai}B_{ai}^T = 0 \tag{2.18}$$

$$A_{ao}^TQ_{ao} + Q_{ao}A_{ao} + C_{ao}^TC_{ao} = 0 \tag{2.19}$$

By expanding $(1,1)$ and $(2,2)$ block of the above equations, we get the following Lyapunov equations

$$AP_e + P_eA^T + X_{ens} = 0 \tag{2.20}$$

$$A^TQ_e + Q_eA + Y_{ens} = 0 \tag{2.21}$$

where X_{ens} and Y_{ens} are

$$X_{ens} = BC_{in}P_{12}^T + P_{12}C_{in}^T B^T + BD_{in}D_{in}^T B^T \tag{2.22}$$

$$Y_{ens} = C^T B_{ou}^T Q_{12}^T + Q_{12}B_{ou}C + C^T D_{ou}^T D_{ou}C \tag{2.23}$$

T_b is the contradient transformation obtained by using P_e and Q_e as

$$T_b^T Q_e T_b = T_b^{-1} P_e T_b^{-T} = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & \sigma_n \end{bmatrix}$$

where $\sigma_j \geq \sigma_{j+1}$, $j = 1, 2, ..., n-1$, and $\sigma_k > \sigma_{k+1}$ The realization of original stable system is transformed into an internally balanced realization by using the contragradient transformation matrix T_b as

$$\left[\begin{array}{c|c} A_b & B_b \\ \hline C_b & D_b \end{array}\right] = \left[\begin{array}{c|c} T_b^{-1}AT_b & T_b^{-1}B \\ \hline CT_b & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \tag{2.24}$$

ROM is obtained by truncating the least controllable and observable states of balanced realization. The ROM obtained is given by

$$G_{kk}(s) = C_1(sI - A_{11})B_1 + D$$

where $A_{11} \epsilon R^{k \times k}$

Remark 1 *For input weighting case only, the contragradient transformation matrix T_b is obtained using P_e and Q_o. Similarly, for output weighting case only, the contragradient transformation matrix T_b is obtained using P_i and Q_e, where P_i and Q_o are the unweighed Gramians enumerated as:*

$$AP_e + P_e A^T + X_{ens} = 0$$

$$A^T Q_e + Q_e A + Y_{ens} = 0$$

Remark 2 *The technique does not give the assurance of the stability of ROMs in double-sided weighting case because it is not guaranteed to ensure $X_{ens} \geq 0$ and $Y_{ens} \geq 0$.*

The main drawback of this technique is that it gives unstable ROM in case of double sided weighting. To tackle with this issue of instability many authors have proposed different techniques related to FWMOR. To conquer Enns method [2] disadvantages, Lin and Chiu [7] has proposed an alternate procedure that guarantees stability when two sided weighting are available. In any case, their strategy has a confinement that can work just when entirely appropriate weighting capacity is utilized as a part of and no pole, zero cancelation happens while shaping the augmented system.

These shortcomings of Lin and Chui [7] method were later adjusted by Sreeram et al [8] and VA [6], where [8] summed up [7] to incorporate weights, while [6] holds the dependability of the system not withstanding when shaft zero cancelation happen. VA [6] produces an indistinguishable outcomes from Enns [2] particularly in controller reduction applications. So far controller reduction issue, if Enns [2] system [2] produces unstable ROMs, so does by VA [6] method.

Wang et al's method [9] has likewise solve the stability issue of Enns [2], which not just give stable ROMs within the sight of two sided weighting additionally inferred error bound. The approximation error of Wang et al method [9] was later enhanced by VA [6] as pointed out by Sreeram [10]. This system and its adjustment by VA [6] are realization independent. This implies for a similar unique system , diverse models can be obtained from various acknowledge. IG [24] proposed a frequency weighted model order reduction technique which not only provides stable ROM but also give frequency response error bound. In this technique the matrices X_e and Y_e are made positive definite by subtracting the smallest value from all eigenvalues.

2.4 Frequency Limited Model Order Reduction

2.4.1 Gawronski and Juang's FLMOR technique

GJ [11] proposed a FLMOR technique in which frequency weights are not used but approximation is considered in a limited frequency interval $[\omega_1, \omega_2]$. GJ [11] introduced the frequency limited controllability $P_{f_{GJ}} = P_f(w_2) - P_f(w_1)$ and observability $Q_{f_{GJ}} =$

$Q_f(w_2) - Q_f(w_1)$ Gramians satisfying :

$$AP_{f_{GJ}} + A^T P_{f_{GJ}} + X_{f_{GJ}} = 0 \quad (2.25)$$

$$A^T Q_{f_{GJ}} + Q_{f_{GJ}} A + Y_{f_{GJ}} = 0 \quad (2.26)$$

where

$$X_{f_{GJ}} = (E_f(w_2) - E_f(w_1))BB^T + BB^T(E_f^*(w_2) - E_f^*(w_1)) \quad (2.27)$$

$$Y_{f_{GJ}} = (E_f(w_2) - E_f(w_1))C^T C + C^T C(E_f^*(w_2) - E_f^*(w_1)) \quad (2.28)$$

$$E_f(w) = \frac{j}{2\pi} \ln((j\omega I + A)(-j\omega I + A)^{-1}) \quad (2.29)$$

$$X_{f_{GJ}} = U \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} U^T, Y_{f_{GJ}} = V \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} V^T \quad (2.30)$$

$S_1 = diag(si_1, si_2, \cdots, si_l), S_2 = diag(si_{l+1}, si_{l+2}, \cdots, si_n)$,
$R_1 = diag(ri_1, ri_2, \cdots, ri_k), R_2 = diag(ri_{k+1}, ri_{k+2}, \cdots, ri_n)$. $l \leq n$ and $k \leq n$ are the number of positive eigenvalues of $X_{f_{GJ}}$ and $Y_{f_{GJ}}$ respectively. Let

$$T_f^T Q_{f_{GJ}} T_f = T_f^{-1} P_{f_{GJ}} T_f^{-T} = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & \sigma_n \end{bmatrix}$$

T_f is a contragredient transformation matrix which is used to transform the original system realization into an internally balanced realization. where $\sigma_h \geq \sigma_{h+1},\ h = 1, 2, \dots, n-1$. Calculation of ROMs are done by segregating the transformed realiation.

Remark 3 *For approximation, multiple frequency intervals can be considered. For example, for two intervals* $[\omega_1, \omega_2]$ *and* $[\omega_3, \omega_4]$, $\omega_1 < \omega_2$, $\omega_3 < \omega_4$.

Remark 4 *Since the symmetric matrices* $X_{f_{GJ}}$ *and* $Y_{f_{GJ}}$ *may becomes indefinite, the ROMs obtained by GJ may not be stable [12].*

2.4.2 Gugercin and Antoulas's FLMOR technique

The stability issue of GJ [11] was highlighted by GA [12]. GA [12] introduced the frequency limited controllability $P_{f_{GA}} = P_f(w_2) - P_f(w_1)$ and observability $Q_{f_{GA}} = Q_f(w_2) - Q_f(w_1)$ Gramians satisfying the following Lyapunov equations :

$$AP_{f_{GA}} + A^T P_{f_{GA}} + X_{f_{GA}} = 0 \tag{2.31}$$

$$A^T Q_{f_{GA}} + Q_{f_{GA}} A + Y_{f_{GA}} = 0 \tag{2.32}$$

The matrices $B_{f_{GA}}$ and $C_{f_{GA}}$ are the updated input and output matrices respectively defined as:

$$B_{f_{GA}} = U_{f_{GA}} |S_{f_{GA}}|^{\frac{1}{2}} \tag{2.33}$$

$$C_{f_{GA}} = |R_{f_{GA}}|^{\frac{1}{2}} V_{f_{GA}}^T \tag{2.34}$$

The expressions $U_{f_{GA}}$, $S_{f_{GA}}$, $V_{f_{GA}}$ and $R_{f_{GA}}$ are obtained as

$$X_{f_{GJ}} = U_{f_{GA}} S_{f_{GA}} U_{f_{GA}}^T \tag{2.35}$$

$$Y_{f_{GJ}} = V_{f_{GA}} R_{f_{GA}} V_{f_{GA}}^T \tag{2.36}$$

where

$$S_{f_{GA}} = \begin{bmatrix} s_{i_1} & 0 & \dots & 0 \\ 0 & s_{i_2} & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & s_{i_n} \end{bmatrix}, R_{f_{GA}} = \begin{bmatrix} r_{i_1} & 0 & \dots & 0 \\ 0 & r_{i_2} & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & r_{i_n} \end{bmatrix}$$

where $si_1 \geq si_2 \geq \cdots \geq si_n$, and $ri_1 \geq ri_2 \geq \cdots \geq ri_n$. Let

$$T_f^T Q_{f_{GA}} T_f = T_f^{-1} P_{f_{GA}} T_f^{-T} = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & \sigma_n \end{bmatrix}$$

To transform the original model to ROMs, T_f is a contragradient transformation matrix where $\sigma_h \geq \sigma_{h+1},\ h = 1, 2, \dots, n-1$. T_f is a transformation used to transform the original system realization into an internally balanced realization. Calculation of ROMs are done by segregating the transformed realiation.

Remark 5 *In this case* $X_{f_{GJ}} \leq B_{f_{GA}} B_{f_{GA}}^T \geq 0, Y_{f_{GJ}} \leq C_{f_{GA}}^T C_{f_{GA}} \geq 0, P_{f_{GA}} > 0 and Q_{f_{GA}} > 0$*, the minimality and stability of* $\{A, B_{f_{GA}}, C_{f_{GA}}\}$ *is guaranteed. Moreover this technique additionally produce frequency response error bounds*

2.4.3 Ghafoor and Sreeram FLMOR technique

GS [4] likewise addresses the stability issue of GJ [11] method. GS [4] ensured the positive definiteness of the symmetric matrices by retaining the positive eigenvalues of the symmetric matrices $X_{f_{GJ}}$ and $Y_{f_{GJ}}$ and truncating the negative eigenvalues. GS [4] introduced the frequency limited controllability $P_{f_{GS}} = P_f(w_2) - P_f(w_1)$ and observability $Q_{f_{GS}} = Q_f(w_2) - Q_f(w_1)$ Gramians satisfying :

$$AP_{f_{GS}} + A^T P_{f_{GS}} + X_{f_{GS}} = 0 \tag{2.37}$$

$$A^T Q_{f_{GS}} + Q_{f_{GS}} A + Y_{f_{GS}} = 0 \tag{2.38}$$

The matrices $B_{f_{GS}}$ and $C_{f_{GS}}$ are the updated input and output matrices respectively described as:

$$B_{f_{GS}} = U_{f_{GS}} |S_{f_{GS}}|^{\frac{1}{2}} \tag{2.39}$$

$$C_{f_{GS}} = |R_{f_{GS}}|^{\frac{1}{2}} V_{f_{GS}}^T \tag{2.40}$$

$$X_{f_{GJ}} = \begin{bmatrix} U_{GS_1} & U_{GS_2} \end{bmatrix} \begin{bmatrix} S_{GS_1} & 0 \\ 0 & S_{GS_2} \end{bmatrix} \begin{bmatrix} U_{GS_1}^T \\ U_{GS_2}^T \end{bmatrix} \tag{2.41}$$

$$Y_{f_{GA}} = \begin{bmatrix} V_{GS_1} & V_{GS_2} \end{bmatrix} \begin{bmatrix} R_{GS_1} & 0 \\ 0 & R_{GS_2} \end{bmatrix} \begin{bmatrix} V_{GS_1}^T \\ V_{GS_2}^T \end{bmatrix} \tag{2.42}$$

where

$$\begin{bmatrix} S_{GS_1} & 0 \\ 0 & S_{GS_2} \end{bmatrix} = diag\{si_1, si_2, ..., si_n\}, \begin{bmatrix} R_{GS_1} & 0 \\ 0 & R_{GS_2} \end{bmatrix} = diag\{ri_1, ri_2, ri_3, ..., ri_n\}$$

$$si_1 \geq si_2 \geq si_3 \geq ... \geq si_n, ri_1 \geq ri_2 \geq r_3 \geq ... \geq r_n$$

$$S_{GS_1} = diag\{s_1, s_2, s_3, ..., s_e\}, R_{GS_1} = diag\{ri_1, ri_2, ri_3, ..., ri_e\}$$

$$si_1 \geq si_2 \geq si_3 \geq ... \geq si_e \geq 0, ri_1 \geq ri_2 \geq ri_3 \geq ... \geq ri_e \geq 0$$

Let

$$T_f^T Q_{f_{GS}} T_f = T_f^{-1} P_{f_{GS}} T_f^{-T} = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & \sigma_n \end{bmatrix}$$

Original stable system is transformed into an internally balanced realization by using contragredient transformation matrix T_f where $\sigma_j \geq \sigma_{j+1},\ j = 1, 2, \dots, n-1$. Calculation of ROMs is done by segregating the least controllable and least observable states of the transformed realization.

Remark 6 *In this case* $X_{f_{GJ}} \leq B_{f_{GS}} B_{f_{GS}}^T \geq 0, Y_{f_{GJ}} \leq C_{f_{GS}}^T C_{f_{GS}} \geq 0, P_{f_{GS}} > 0$ *and* $Q_{f_{GS}} > 0$*, the minimality and stability of* $\{A, B_{f_{GS}}, C_{f_{GS}}\}$ *is guaranteed. Moreover this technique also give frequency response error bounds*

2.4.4 Imran and Ghafoor's FLMOR technique

In GA technique [12] , the symmteric matrices $X_{f_{GJ}}$ and $Y_{f_{GJ}}$ are guaranteed positive definite/semipositive definite respectively by taking the square root of absolutes values estimations of the eigenvalues by eigenvalue decomposition (EVD) of symmetric $X_{f_{GJ}}$ and $Y_{f_{GJ}}$. This occasionally prompts to a substantial change in some eigenvalues and may not impact other eigen values. Then again , GS [4] guarantees positive definitness of the matrices $X_{f_{GJ}}$ and $Y_{f_{GJ}}$ by effecting just positive eigenvalues and by replacing negative eigenvalues with zeros. This system likewise doesnot have comparative impact on all eigenvalues. In IG [13] a technique is proposed where exertion is to similarly affect all eigenvalues of uncertain matrices $X_{f_{GJ}}$ and $Y_{f_{GJ}}$. The ROMs got are ensured to be stable . Additionally, it has error bounds and enhanced frequency response error. Take new controlability $P_{f_{IG}}$ and Observability $Q_{f_{IG}}$ Gramians respectively, are determined by resolving the following Lyapunov equations:

$$AP_{f_{IG}} + P_{f_{IG}} A^T + X_{f_{IG}} = 0 \tag{2.43}$$

$$A^T Q_{f_{IG}} + Q_{f_{IG}} A + Y_{f_{IG}} = 0 \tag{2.44}$$

The matrices $B_{f_{IG}}$ and $C_{f_{IG}}$ are new updated input and output matrices respectively defined as :

$$B_{f_{IG}} = \begin{cases} U_{f_{IG}}(S_{f_{IG}} - si_n I)^{1/2} & \text{for } si_n < 0 \\ U_{f_{IG}} S_{f_{IG}}^{1/2} & \text{for } si_n \geq 0 \end{cases} \tag{2.45}$$

$$C_{f_{IG}} = \begin{cases} (R_{f_{IG}} - ri_n I)^{1/2} V_{f_{IG}}^T & \text{for } r_n < 0 \\ R_{f_{IG}}^{1/2} V_{f_{IG}}^T & \text{for } ri_n \geq 0. \end{cases} \tag{2.46}$$

The terms $U_{f_{IG}}, S_{f_{IG}}, V_{f_{IG}}$, and $R_{f_{IG}}$ are solved as

$$X_{f_{GJ}} = U_{f_{IG}} S_{f_{IG}} U_{f_{IG}}^T \tag{2.47}$$

$$Y_{f_{GJ}} = V_{f_{IG}} R_{f_{IG}} V_{f_{IG}}^T \tag{2.48}$$

where

$$S_{f_{IG}} = \begin{bmatrix} s_{i_1} & 0 & \dots & 0 \\ 0 & s_{i_2} & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & s_{i_n} \end{bmatrix}, R_{f_{IG}} = \begin{bmatrix} r_{i_1} & 0 & \dots & 0 \\ 0 & r_{i_1} & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & r_{i_n} \end{bmatrix}$$

where $si_1 \geq si_2 \geq \cdots \geq si_n$, and $ri_1 \geq ri_2 \geq \cdots \geq ri_n$. A consideration is made that to transform a original system into an internally balanced realization. Transformation matrix T_f is obtained using the controllability Gramian $P_{f_{IG}}$ and observability Gramians $Q_{f_{IG}}$ as Let

$$T_f^T Q_{f_{IG}} T_f = T_f^{-1} P_{f_{IG}} T_f^{-T} = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & \sigma_n \end{bmatrix}$$

Determination of ROMs is carried out by segregating the transformed realization where $\sigma_h \geq \sigma_{h+1}, \ h = 1, 2, 3, \dots, n-1, \sigma_l > \sigma_{l+1}$.

Remark 7 *Since $X_{f_{GJ}} \leq B_{f_{IG}} B_{f_{IG}}^T$, $Y_{f_{GJ}} \leq C_{f_{IG}}^T C_{f_{IG}}$, $B_{f_{IG}} B_{f_{IG}}^T \geq 0$, $C_{f_{IG}}^T C_{f_{IG}} \geq 0$, $P_{f_I G} > 0$ and $Q_{f_{IG}} > 0$. Consequently, the realization $(A, B_{f_{IG}}, C_{f_{IG}})$ is minimal and also this technique gives the assurance of the stability of ROMs.*

Theorem 1 *In IG [13] technique, the following error bound formula holds provided that* *the following rank conditions* $rank\,[B_{f_{IG}}\;\; B] = rank\,[B_{f_{IG}}]$ *and* $rank\begin{bmatrix} C_{f_{IG}} \\ C \end{bmatrix} = rank\,[C_{f_{IG}}]$ *(which follows from [25]) are satisfied*

$$\|G(s) - G_{kk}(s)\|_\infty \leq 2\|L_{f_{IG}}\|\|K_{f_{IG}}\|\sum_{h=l+1}^{n}\sigma_h$$

where

$$L_{f_{IG}} = \begin{cases} CV_{f_{IG}}(R_{f_{IG}} - ri_n I)^{-1/2} & \textit{for } ri_n < 0 \\ CV_{f_{IG}}R_{f_{IG}}^{-1/2} & \textit{for } ri_n \geq 0 \end{cases} \tag{2.49}$$

$$K_{f_{IG}} = \begin{cases} (S_{f_{IG}} - si_n I)^{-1/2}U_{IG}^T B & \textit{for } si_n < 0 \\ S_{f_{IG}}^{-1/2}U_{IG}^T B & \textit{for } si_n \geq 0 \end{cases} \tag{2.50}$$

Proof: Since $rank\,[B_{f_{IG}}\;\; B] = rank\,[B_{f_{IG}}]$ and $rank\begin{bmatrix} C_{f_{IG}} \\ C \end{bmatrix} = rank\,[C_{f_{IG}}]$, the relationships $B = B_{f_{IG}}K_{f_{IG}}$ and $C = L_{f_{IG}}C_{f_{IG}}$ hold. By partitioning $B_{f_{IG}} = \begin{bmatrix} B_{f_{IG_1}} \\ B_{f_{IG_2}} \end{bmatrix}$, $C_{f_{IG}} = \begin{bmatrix} C_{f_{IG_1}} & C_{f_{IG_2}} \end{bmatrix}$ and substituting $B_1 = B_{f_{IG_1}}K_{f_{IG}}$, $C_1 = L_{f_{IG}}C_{f_{IG_1}}$ respectively produces

$$\begin{aligned} \|G(s) - G_{kk}(s)\|_\infty &= \|C(sI-A)^{-1}B - C_1(sI-A_{11})^{-1}B_1\|_\infty \\ &= \|L_{f_{IG}}C_{f_{IG}}(sI-A)^{-1}B_{f_{IG}}K_{f_{IG}} - L_{f_{IG}}C_{f_{IG_1}}(sI-A_{11})^{-1}B_{f_{IG_1}}K_{f_{IG}}\|_\infty \\ &= \|L_{f_{IG}}(C_{f_{IG}}(sI-A)^{-1}B_{f_{IG}} - C_{f_{IG_1}}(sI-A_{11})^{-1}B_{f_{IG_1}})K_{f_{IG}}\|_\infty \\ &\leq \|L_{f_{IG}}\|\|(C_{f_{IG}}(sI-A)^{-1}B_{f_{IG}} - C_{f_{IG_1}}(sI-A_{11})^{-1}B_{f_{IG_1}})\|_\infty\|K_{f_{IG}}\| \end{aligned}$$

If $\{A_{11}, B_{f_{IG_1}}, C_{f_{IG_1}}\}$ is ROM obtained by segregating a balanced realization $\{A, B_{f_{IG}}, C_{f_{IG}}\}$, we have from [2]

$$\|(C_{f_{IG}}(sI-A)^{-1}B_{f_{IG}} - C_{f_{IG_1}}(sI-A_{11})^{-1}B_{f_{IG_1}})\|_\infty \leq 2\sum_{h=l+1}^{n}\sigma_h.$$

Therefore,

$$\|G(s) - G_{kk}(s)\|_\infty \leq 2\|L_{f_{IG}}\|\|K_{f_{IG}}\|\sum_{h=l+1}^{n}\sigma_h$$

Remark 8 *When the matrices are symmetric* $X_{f_{GJ}} \geq 0$ *and* $Y_{f_{GJ}} \geq 0$*, therefore* $P_{f_{GJ}} = P_{f_{IG}}$ *and* $Q_{f_{GJ}} = Q_{f_{IG}}$*. Otherwise* $P_{f_{GJ}} < P_{f_{IG}}$ *and* $Q_{f_{GJ}} < Q_{f_{IG}}$*. In addition, Hankel singular values satisfies:* $(\lambda_j[P_{f_{GJ}}Q_{f_{GJ}}])^{1/2} \leq (\lambda_j[P_{f_{IG}}Q_{f_{IG}}])^{1/2}$.

2.5 Time Limited Model Order Reduction

Many real MOR problems are naturally time dependent. Mostly the response of the system is more important in a particular time interval rather than over the whole time range. TLMOR found their nearness in various applications which incorporate semidiscretization of fractional differential conditions, multibody elements with requirements, electrical circuit recreation and small scale electro-mechanical framework, parameter evaluation with a particular attention to real-time computing in biomedical engineering and computational physics, the study of high-dimensional problems in state space, physical space. In order to conquer these real world problems, GJ [11] presented a TLMOR technique in which the new controllability and observability Gramains are defined over a limited time interval. But the disadvantage of GJ [11] technique is that the stability of ROM is not ensured [12]. GA [12] proposed a TLMOR technique in which he addressed the instability issue. ROM obtained using GA [12] technique is stable but it leads to large approximation error. In TLMOR the system response is measured in the time interval $T = [t_1, t_2], t_2 > t_1 \geq 0$. The time limited Gramians are defined over the time interval $T = [t_1, t_2]$. GJ [11] produced less approximation error but ROM obtained using this technique is unstable. MI [17] presented two TLMOR techniques. Both techniques produces stable ROM but yields large error. Additionally MI [17] also gives error bound.

2.5.1 Gawronski and Juang's TLMOR technique

GJ [11] proposed a TLMOR technique, which estimates the original system (in the limited time interval, $[t_1, t_2]$, $[t_2 > t_1]$. GJ [11] defined the time limited controllability $P_{t_{GJ}}$ and observability $Q_{t_{GJ}}$ Gramians satisfying :

$$AP_{t_{GJ}} + P_{t_{GJ}}A^T + X_{t_{GJ}} = 0 \quad (2.51)$$

$$A^TQ_{t_{GJ}} + Q_{t_{GJ}}A + Y_{t_{GJ}} = 0 \quad (2.52)$$

These time limited Gramians are defined as

$$P_{t_{GJ}} = \int_{t_1}^{t_2} e^{A\tau} BB^T e^{A^T\tau} d\tau \tag{2.53}$$

$$Q_{t_{GJ}} = \int_{t_1}^{t_2} e^{A^T\tau} C^T C e^{A\tau} d\tau \tag{2.54}$$

These Gramians are determined by the following equations

$$P_{t_{GJ}} = P_c(t_1) - P_c(t_2) \tag{2.55}$$

$$Q_{t_{GJ}} = Q_o(t_1) - Q_o(t_2) \tag{2.56}$$

where

$$P_c(t) = S_{GJ}(t) P S_{GJ}^T(t) \tag{2.57}$$

$$Q_o(t) = S_{GJ}(t)^T Q S_{GJ}(t) \tag{2.58}$$

$$S_{GJ}(t) = e^{A\tau} \tag{2.59}$$

The equations (2.55) and (2.56) are determined as follows.

Let

$$\theta_{GJ}(t) = \int_0^t e^{A\tau} BB^T e^{A^T\tau} \tag{2.60}$$

As given in Kailath [26]

$$\theta_{GJ}(t) = P - S_{GJ}(t) P S_{GJ}^T(t) \tag{2.61}$$

$$= P - P_c(t) \tag{2.62}$$

As $P_{t_{GJ}}$ can be written as

$$P_{t_{GJ}} = \theta_{GJ}(t_1) - \theta_{GJ}(t_2) \tag{2.63}$$

By putting equation (2.62) in equation (2.63) we get

$$P_{t_{GJ}} = P_c(t_1) - P_c(t_2) \tag{2.64}$$

Similarly we can obtain

$$Q_{t_{GJ}} = Q_o(t_1) - Q_o(t_2) \tag{2.65}$$

Denote

$$X_c(t) = S_{GJ}(t)BB^T S_{GJ}^T(t) \tag{2.66}$$

$$Y_o(t) = S_{GJ}^T(t)C^T C S_{GJ}(t) \tag{2.67}$$

Symmetric matrices can be calculated using given relationship as

$$X_{t_{GJ}} = X_c(t_1) - X_c(t_2) \tag{2.68}$$

$$Y_{t_{GJ}} = Y_o(t_1) - Y_o(t_2) \tag{2.69}$$

$$X_{t_{GJ}} = e^{At_1}BB^T e^{A^T t_1} - e^{At_2}BB^T e^{A^T t_2} \tag{2.70}$$

$$Y_{t_{GJ}} = e^{A^T t_1}C^T C e^{At_1} - e^{A^T t_2}C^T C e^{At_2} \tag{2.71}$$

$$X_{t_{GJ}} = \begin{bmatrix} U1 & U2 \end{bmatrix} \begin{bmatrix} S1 & 0 \\ 0 & S2 \end{bmatrix} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} \tag{2.72}$$

$$Y_{t_{GJ}} = \begin{bmatrix} V1 & V2 \end{bmatrix} \begin{bmatrix} R1 & 0 \\ 0 & R2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \tag{2.73}$$

$$S_1 = \begin{bmatrix} s_1 & 0 & \dots & 0 \\ 0 & s_2 & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & s_l \end{bmatrix}, S_2 = \begin{bmatrix} s_{l+1} & 0 & \dots & 0 \\ 0 & s_{l+2} & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & s_n \end{bmatrix}$$

$$R_1 = \begin{bmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & r_m \end{bmatrix}, R_2 = \begin{bmatrix} r_{m+1} & 0 & \dots & 0 \\ 0 & r_{m+2} & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & r_n \end{bmatrix}$$

where $l \le n$ and $m \le n$ are the number of positive eigenvalues of the symmetric matrices $X_{t_{GJ}}$ and $Y_{t_{GJ}}$ respectively. The contradient transformation matrix T_t is determined using the time limited controllability and observability Gramians $P_{t_{GJ}}$ and $Q_{t_{GJ}}$ respectively as

$$T_t^T Q_{t_{GJ}} T_t = T_t^{-1} P_{t_{GJ}} T_t^{-T} = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & \sigma_n \end{bmatrix}$$

The realization of original stable system is transformed into an internally balanced realization by using the contragradient transformation matrix T_t as

$$\left[\begin{array}{c|c} A_b & B_b \\ \hline C_b & D_b \end{array}\right] = \left[\begin{array}{c|c} T_t^{-1} A T_t & T_t^{-1} B \\ \hline C T_t & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \tag{2.74}$$

ROM is obtained by segregating the least controllable and observable states of balanced realization. The ROM obtained is given by

$$G_{t_{kk}}(s) = C_1(sI - A_{11})B_1 + D$$

where $A_{11} \epsilon R^{k \times k}$

Remark 9 *The symmetric matrices $X_{t_{GJ}}$ and $Y_{t_{GJ}}$ are not guaranteed to be positive definite, the ROM obtained using GJ [11] technique may not be stable [12].*

2.5.2 Gugercin and Antoulas's TLMOR technique [12]

GA [12] highlighted the stability issue of GJ [11]. GA [12] introduced the time limited controllability and observability Gramians $P_{t_{GA}}$ and $Q_{t_{GA}}$ fulfilling the following Lyapunov equations :

$$AP_{t_{GA}} + P_{t_{GA}} A^T + X_{t_{GA}} = 0 \tag{2.75}$$

$$A^T Q_{t_{GA}} + Q_{t_{GA}} A + Y_{t_{GA}} = 0 \tag{2.76}$$

In GJ [11] technique the matrices $X_{t_{GJ}}$ and $Y_{t_{GJ}}$ may become indefinite, this is the main reason of instability of the ROM obtained by GJ [11]. GA [12] ensured the positive definitiveness by taking the absolute of the eigenvalues of the matrices $X_{t_{GJ}}$ and $Y_{t_{GJ}}$. The updated input and output matrices $B_{t_{GA}}$ and $C_{t_{GA}}$ are defined as

$$B_{t_{GA}} = U_{t_{GA}} |S_{t_{GA}}|^{\frac{1}{2}} \tag{2.77}$$

$$C_{t_{GA}} = |R_{t_{GA}}|^{\frac{1}{2}} V_{t_{GA}}^T \tag{2.78}$$

where $U_{t_{GA}}$, $S_{t_{GA}}$, $V_{t_{GA}}$ and $R_{t_{GA}}$ are the terms obtained from the singular value decomposition of $X_{t_{GJ}}$ and $Y_{t_{GJ}}$ as

$$X_{t_{GJ}} = U_{t_{GA}} S_{t_{GA}} U_{t_{GA}}^T \tag{2.79}$$

$$Y_{t_{GJ}} = V_{t_{GA}} R_{t_{GA}} V_{t_{GA}}^T \tag{2.80}$$

The updated symmetric matrices $X_{t_{GA}}$ and $Y_{t_{GA}}$ is obtained from equ (2.77) and (2.78).

$$X_{t_{GA}} = B_{t_{GA}} B_{t_{GA}}^T \tag{2.81}$$

$$Y_{t_{GA}} = C_{t_{GA}}^T C_{t_{GA}} \tag{2.82}$$

where

$$S_{t_{GA}} = \begin{bmatrix} s_1 & 0 & \dots & 0 \\ 0 & s_2 & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & s_n \end{bmatrix}, R_{t_{GA}} = \begin{bmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & r_n \end{bmatrix}$$

$|s_1| \geq |s_2| \geq ... \geq |s_n| \geq 0$ and $|r_1| \geq |r_2| \geq ... \geq |r_n| \geq 0$. The contradient transformation matrix T_t is determined using the new controllability and observability Gramians $P_{t_{GA}}$ and $Q_{t_{GA}}$ respectively such that

$$T_t^T Q_{t_{GA}} T_t = T_t^{-1} P_{t_{GA}} T_t^{-T} = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & \sigma_n \end{bmatrix}$$

The realization of original stable system is transformed into an internally balanced realization by using the contragradient transformation matrix T_t as

$$\left[\begin{array}{c|c} A_b & B_b \\ \hline C_b & D_b \end{array}\right] = \left[\begin{array}{c|c} T_t^{-1}AT_t & T_t^{-1}B \\ \hline CT_t & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \tag{2.83}$$

ROM is obtained by segregating the least controllable and observable states of balanced realization. The ROM obtained is given by

$$G_{t_{kk}}(s) = C_1(sI - A_{11})B_1 + D$$

where $A_{11}\epsilon R^{k\times k}$

Remark 10 *In this case* $X_{t_{GJ}} \leq B_{t_{GA}}B_{t_{GA}}^T \geq 0, Y_{t_{GJ}} \leq C_{t_{GA}}^T C_{t_{GA}} \geq 0$*,* $P_{t_{GA}} > 0$ *and* $Q_{t_{GA}} > 0$*, it is guaranteed that* $\{A, B_{t_{GA}}, C_{t_{GA}}\}$ *is minimal and stable. GA [12] also give frequency response error bounds.*

2.5.3 Imran TLMOR technique I [17]

Motivated from GS [4], MI [17] modified the GJ [11] to obtain stable ROM. In this technique the symmetric matrices $X_{t_{GJ}}$ and $Y_{t_{GJ}}$ are made positive definitive by accomplishing the EVD. The positive eigenvalues of the symmetric matrices $X_{t_{GJ}}$ and $Y_{t_{GJ}}$ are retained and the negative eigenvalues are truncated. Let new controllability P_{t_G} and observability Q_{t_G} Gramians satisfying:

$$AP_{t_G} + P_{t_G}A^T + B_{t_G}B_{t_G}^T = 0 \tag{2.84}$$

$$A^TQ_{t_G} + Q_{t_G}A + C_{t_G}^TC_{t_G} = 0 \tag{2.85}$$

The updated input and output matrices B_{t_G} and C_{t_G} are defined as:

$$B_{t_G} = \begin{cases} U_{t_G} \begin{bmatrix} S_{t_{G_1}} & 0 \\ 0 & 0 \end{bmatrix}^{1/2} & \text{for } s_n < 0 \\ U_{t_G}(S_{t_G})^{1/2} & \text{for } s_n \geq 0 \end{cases} \tag{2.86}$$

$$C_{t_G} = \begin{cases} \begin{bmatrix} R_{t_{G_1}} & 0 \\ 0 & 0 \end{bmatrix} R_{t_G}^T & \text{for } r_n < 0 \\ (R_{t_G})^{1/2} V_{t_G}^T & \text{for } r_n \geq 0 \end{cases} \tag{2.87}$$

The expressions U_{t_G}, S_{t_G}, R_{t_G} and V_{t_G} are determined by the SVD of the symmetric matrices $X_{t_{GJ}}$ and $Y_{t_{GJ}}$ as

$$\begin{aligned} X_{t_{GJ}} &= U_{t_G} S_{t_G} U_{t_G}^T \\ X_{t_{GJ}} &= V_{t_G} R_{t_G} V_{t_G}^T \end{aligned}$$

where

$$U_{t_G} = \begin{bmatrix} U_{t_{G_1}} & U_{t_{G_2}} \end{bmatrix}, S_{t_G} = \begin{bmatrix} S_{t_{G_1}} & 0 \\ 0 & S_{t_{G_2}} \end{bmatrix}$$

$$V_{t_G} = \begin{bmatrix} V_{t_{G_1}} & V_{t_{G_2}} \end{bmatrix}, R_{t_G} = \begin{bmatrix} R_{t_{G_1}} & 0 \\ 0 & R_{t_{G_2}} \end{bmatrix}$$

where

$$\begin{bmatrix} S_{t_{G_1}} & 0 \\ 0 & S_{t_{G_2}} \end{bmatrix} = \begin{bmatrix} s_{g_1} & 0 & \dots & 0 \\ 0 & s_{g_2} & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & s_{g_n} \end{bmatrix}$$

$$\begin{bmatrix} R_{t_{G_1}} & 0 \\ 0 & R_{t_{G_2}} \end{bmatrix} = \begin{bmatrix} r_{g_1} & 0 & \dots & 0 \\ 0 & r_{g_2} & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & r_{g_n} \end{bmatrix}$$

$$S_{t_{G_1}} = diag(s_{g_1}, s_{g_2}, \cdots, s_{g_l}), S_{t_{G_2}} = diag(s_{g_{l+1}}, s_{g_{l+2}}, \cdots, r_{g_n})$$

$$R_{t_{G_1}} = diag(r_{g_1}, r_{g_2}, \cdots, r_{g_l}), R_{t_{G_2}} = diag(r_{g_{l+1}}, r_{g_{l+2}}, \cdots, r_{g_n})$$

where $s_{r_1} \geq s_{r_2} \geq ... \geq s_{r_n}$ and $r_{r_1} \geq r_{r_2} \geq ... \geq r_{r_n}$ Let the contragradient transformation matrix T_{t_G} be obtained using the new controllability and observability Gramians P_{t_G} and Q_{t_G} respectively.

$$T_{t_G}^T Q_{t_G} T_{t_G} = T_{t_G}^{-1} P_{t_G} T_{t_G}^{-T} = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & \sigma_n \end{bmatrix}$$

where $\sigma_j \geq \sigma_{j+1}, j = 1, 2, 3, ..., n-1$. Contragradient transformation T_{t_G} is applied to the original system to get an internally balanced realization.

$$\left[\begin{array}{c|c} A_b & B_b \\ \hline C_b & D_b \end{array}\right] = \left[\begin{array}{c|c} T_{t_G}^{-1} A T_{t_G} & T_{t_G}^{-1} B \\ \hline C T_{t_G} & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \tag{2.88}$$

ROM is achieved by segregating the least controllable and least observable states of the balanced realization.

$$G_{kk}(s) = C_1(sI - A_{11})^{-1} B_1 + D_1 \tag{2.89}$$

where $\{A_{11}, B_1, C_1, D_1\}$ is the k^{th} $(k < n)$ order minimal realization.

Remark 11 *Since $X_{t_{GJ}} \leq B_{t_G} B_{t_G}^T \leq B_{t_{GA}} B_{t_{GA}}^T$, $Y_{t_{GJ}} \leq C_{t_G} C_{t_G}^T \leq C_{t_{GA}} C_{t_{GA}}^T$ and the realization $\{A, B_{t_G}, C_{t_G}\}$ is minimal, therefore the ROM obtained is stable.*

Theorem 2 *In this technique the following error bound formula holds provided that the following rank conditions $rank\left[B_{t_G} \;\; B\right] = rank\left[B_{t_G}\right]$ and $rank \begin{bmatrix} C_{t_G} \\ C \end{bmatrix} = rank\left[C_{t_G}\right]$ (which follows from [25]) are satisfied*

$$\|G(s) - G_{kk}(s)\|_\infty \leq 2\|L_{t_G}\|\|K_{t_G}\| \sum_{m=l+1}^{n} \sigma_m$$

where $L_{t_G} = C V_{t_{G_1}} R_{t_{G_1}}^{-1/2}$ and $K_{t_G} = S_{t_{G_1}}^{-1/2} U_{t_{G_1}}^T B$

Proof: Since $rank\,[B_{t_G}\ \ B] = rank\,[B_{t_G}]$ and $rank \begin{bmatrix} C_{t_G} \\ C \end{bmatrix} = rank\,[C_{t_G}]$, the relationships $B = B_{t_G}K_{t_G}$ and $C = L_{t_G}C_{t_G}$ hold. By partitioning $B_{t_G} = \begin{bmatrix} B_{t_{G_1}} \\ B_{t_{G_2}} \end{bmatrix}, C_{t_G} = \begin{bmatrix} C_{t_{G_1}} & C_{t_{G_2}} \end{bmatrix}$ and substituting $B_1 = B_{t_{G_1}}K_{t_G}$, $C_1 = L_{t_G}C_{t_{G_1}}$ respectively produces

$$\begin{aligned}
\|G(s) - G_{kk}(s)\|_\infty &= \|C(sI-A)^{-1}B - C_1(sI-A_{11})^{-1}B_1\|_\infty \\
&= \|L_{t_G}C_{t_G}(sI-A)^{-1}B_{t_G}K_{t_G} - L_{t_G}C_{t_{G_1}}(sI-A_{11})^{-1}B_{t_{G_1}}K_{t_G}\|_\infty \\
&= \|L_{t_G}(C_{t_G}(sI-A)^{-1}B_{t_G} - C_{t_{G_1}}(sI-A_{11})^{-1}B_{t_{G_1}})K_{t_G}\|_\infty \\
&\leq \|L_{t_G}\|\|(C_{t_G}(sI-A)^{-1}B_{t_G} - C_{t_{G_1}}(sI-A_{11})^{-1}B_{t_{G_1}})\|_\infty\|K_{t_G}\|
\end{aligned}$$

If $\{A_{11}, B_{t_{G_1}}, C_{t_{G_1}}\}$ is ROM achieved by segregating a balanced system $\{A, B_{t_G}, C_{t_G}\}$, we have [2].

$$\|(C_{t_G}(sI-A)^{-1}B_{t_G} - C_{t_{G_1}}(sI-A_{11})^{-1}B_{t_{G_1}})\|_\infty \leq 2\sum_{m=l+1}^{n}\sigma_m.$$

Therefore,

$$\|G(s) - G_{kk}(s)\|_\infty \leq 2\|L_{t_G}\|\|K_{t_G}\| \sum_{m=l+1}^{n}\sigma_m$$

2.5.4 Imran TLMOR technique II [17]

MI [17] proposed another TLMOR technique (Motivated from [13]). In this technique exertion is to similarly affect all eigenvalues of uncertain matrices $X_{t_{GJ}}$ and $Y_{t_{GJ}}$. In this technique positive definiteness is ensured by subtracting the smallest eigenvalue value from all eigenvalues of the matrices $X_{t_{GJ}}$ and $Y_{t_{GJ}}$. The stability of the ROM obtained is ensured to be stable. Furthermore frequency response error bound is also provided. Let new controllability P_{t_I} and observability Q_{t_I} Gramians satisfying:

$$AP_{t_I} + P_{t_I}A^T + B_{t_I}B_{t_I}^T = 0 \tag{2.90}$$

$$A^TQ_{t_I} + Q_{t_I}A + C_{t_I}^TC_{t_I} = 0 \tag{2.91}$$

The updated input and output matrices B_{t_I} and C_{t_I} respectivelys are defined as:

$$B_{t_I} = \begin{cases} U_{t_I}(S_{t_I} - s_n I)^{1/2} & \text{for } s_n < 0 \\ U_{t_I}(S_{t_I})^{1/2} & \text{for } s_n \geq 0 \end{cases} \tag{2.92}$$

$$C_{t_I} = \begin{cases} (R_{t_I} - r_n I)^{1/2} V_{t_I}^T & \text{for } r_n < 0 \\ (R_{t_I})^{1/2} V_{t_I}^T & \text{for } r_n \geq 0. \end{cases} \tag{2.93}$$

The expressions U_{t_I}, S_{t_I}, R_{t_I} and V_{t_I} are determined by the SVD of the symmetric matrices $X_{t_{GJ}}$ and $Y_{t_{GJ}}$ as

$$X_{t_{GJ}} = U_{t_I} S_{t_I} U_{t_I}^T \tag{2.94}$$

$$Y_{t_{GJ}} = V_{t_I} R_{t_I} V_{t_I}^T \tag{2.95}$$

where

$$U_{t_I} = \begin{bmatrix} U_{t_{I_1}} & U_{t_{I_2}} \end{bmatrix}, S_{t_I} = \begin{bmatrix} S_{t_{I_1}} & 0 \\ 0 & S_{t_{I_2}} \end{bmatrix}$$

$$V_{t_I} = \begin{bmatrix} V_{t_{I_1}} & V_{t_{I_2}} \end{bmatrix}, R_{t_I} = \begin{bmatrix} R_{t_{I_1}} & 0 \\ 0 & R_{t_{I_2}} \end{bmatrix}$$

where

$$\begin{bmatrix} S_{t_{I_1}} & 0 \\ 0 & S_{t_{I_2}} \end{bmatrix} = \begin{bmatrix} s_{i_1} & 0 & \dots & 0 \\ 0 & s_{i_2} & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & s_{i_n} \end{bmatrix}$$

$$\begin{bmatrix} R_{t_{I_1}} & 0 \\ 0 & R_{t_{I_2}} \end{bmatrix} = \begin{bmatrix} r_{i_1} & 0 & \dots & 0 \\ 0 & r_{i_2} & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & r_{i_n} \end{bmatrix}$$

$$S_{t_{I_1}} = diag(s_{i_1}, s_{i_2}, \cdots, s_{i_l}), S_{t_{I_2}} = diag(s_{i_{l+1}}, s_{i_{l+2}}, \cdots, s_{i_n})$$

$$R_{t_{I_1}} = diag(r_{i_1}, r_{i_2}, \cdots, r_{i_l}), R_{t_{I_2}} = diag(r_{i_{l+1}}, r_{i_{l+2}}, \cdots, r_{i_n})$$

where $s_{i_1} \geq s_{i_2} \geq ... \geq s_{i_n}$ and $r_{i_1} \geq r_{i_2} \geq ... \geq r_{i_n}$ Let the contragradient transformation matrix T_{t_I} be obtained using the new controllability and observability Gramians P_{t_I} and Q_{t_I} respectively.

$$T_{t_I}^T Q_{t_I} T_{t_I} = T_{t_I}^{-1} P_{t_I} T_{t_I}^{-T} = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & \sigma_n \end{bmatrix}$$

where $\sigma_j \geq \sigma_{j+1}, j = 1, 2, 3, ..., n-1$. Contragradient transformation T_{t_I} is applied to the original system to get an internally balanced realization.

$$\left[\begin{array}{c|c} A_b & B_b \\ \hline C_b & D_b \end{array}\right] = \left[\begin{array}{c|c} T_{t_I}^{-1} A T_{t_I} & T_{t_I}^{-1} B \\ \hline C T_{t_I} & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \quad (2.96)$$

ROM is achieved by segregating the least controllable and least observable states of the balanced realization.

$$G_{kk}(s) = C_1(sI - A_{11})^{-1} B_1 + D_1 \quad (2.97)$$

where $\{A_{11}, B_1, C_1, D_1\}$ is the k^{th} $(k < n)$ order minimal realization.

Remark 12 *Since* $X_{t_{GJ}} \leq B_{t_I} B_{t_I}^T \leq B_{t_{GA}} B_{t_{GA}}^T$, $Y_{t_{GJ}} \leq C_{t_I} C_{t_I}^T \leq C_{t_{GA}} C_{t_{GA}}^T$ *and the realization* $\{A, B_{t_I}, C_{t_I}\}$ *is minimal, therefore the ROM obtained is stable.*

Theorem 3 *In this technique the following error bound formula holds provided that the following rank conditions* $rank\,[B_{t_I} \;\; B] = rank\,[B_{t_I}]$ *and* $rank \begin{bmatrix} C_{t_I} \\ C \end{bmatrix} = rank\,[C_{t_I}]$ *(which follows from [25]) are satisfied*

$$\|G(s) - G_{kk}(s)\|_\infty \leq 2\|L_{t_I}\|\|K_{t_I}\| \sum_{m=l+1}^{n} \sigma_m$$

where

$$L_{t_I} = \begin{cases} CV_{t_I}(R_{t_I} - r_n I)^{-1/2} & \textit{for } r_n < 0 \\ CV_{t_I}(R_{t_I})^{-1/2} & \textit{for } r_n \geq 0 \end{cases} \tag{2.98}$$

$$K_{t_I} = \begin{cases} (S_{t_I} - s_n I)^{-1/2} U_{t_I}^T B & \textit{for } s_n < 0 \\ (S_{t_I})^{-1/2} U_{t_I}^T B & \textit{for } s_n \geq 0. \end{cases} \tag{2.99}$$

Proof: Since $rank\,[B_{t_I} \;\; B] = rank\,[B_{t_I}]$ and $rank \begin{bmatrix} C_{t_I} \\ C \end{bmatrix} = rank\,[C_{t_I}]$, the relationships $B = B_{t_I} K_{t_I}$ and $C = L_{t_I} C_{t_I}$ hold. By partitioning $B_{t_I} = \begin{bmatrix} B_{t_{I_1}} \\ B_{t_{I_2}} \end{bmatrix}$, $C_{t_I} = \begin{bmatrix} C_{t_{I_1}} & C_{t_{I_2}} \end{bmatrix}$ and substituting $B_1 = B_{t_{I_1}} K_{t_I}$, $C_1 = L_{t_I} C_{t_{I_1}}$ respectively produces

$$\begin{aligned} \|G(s) - G_{kk}(s)\|_\infty &= \|C(sI - A)^{-1}B - C_1(sI - A_{11})^{-1}B_1\|_\infty \\ &= \|L_{t_I} C_{t_I}(sI - A)^{-1} B_{t_I} K_{t_I} - L_{t_I} C_{t_{I_1}}(sI - A_{11})^{-1} B_{t_{I_1}} K_{t_I}\|_\infty \\ &= \|L_{t_I}(C_{t_I}(sI - A)^{-1} B_{t_I} - C_{t_{I_1}}(sI - A_{11})^{-1} B_{t_{I_1}}) K_{t_I}\|_\infty \\ &\leq \|L_{t_I}\| \|(C_{t_I}(sI - A)^{-1} B_{t_I} - C_{t_{I_1}}(sI - A_{11})^{-1} B_{t_{I_1}})\|_\infty \|K_{t_I}\| \end{aligned}$$

If $\{A_{11}, B_{t_{I_1}}, C_{t_{I_1}}\}$ is ROM achieved by segregating a balanced system realization $\{A, B_{t_I}, C_{t_I}\}$, we have [2].

$$\|(C_{t_I}(sI - A)^{-1} B_{t_I} - C_{t_{I_1}}(sI - A_{11})^{-1} B_{t_{I_1}})\|_\infty \leq 2 \sum_{m=l+1}^{n} \sigma_m.$$

Therefore,

$$\|G(s) - G_{kk}(s)\|_\infty \leq 2\|L_{t_I}\| \|K_{t_I}\| \sum_{m=l+1}^{n} \sigma_m$$

Proposed Techniques

3.1 Proposed Techniques

In this chapter proposed TLMOR techniques are discussed. Existing techniques have issues of stability and large approximation error. GJ [11] technique produces less approximation error but it leads to unstable ROMs. The reason of instability of ROMs is that the symmetric matrices $X_{t_{GJ}}$ and $Y_{t_{GJ}}$ may become indefinite. GA [12] gave an idea of making the matrices $X_{t_{GJ}}$ and $Y_{t_{GJ}}$ positive definite. GA [12] technique ensures the stability of ROM but the drawback of this technique is large approximation error. In GA [12] technique the matrices $X_{t_{GJ}}$ and $Y_{t_{GJ}}$ are made positive definite by taking the absolute of the eigenvalues of the symmetric matrices. This causes a large change in some of the eigenvalues and little effect on rest of eigenvalues. MI I [17] made certain the positive definiteness of the matrices $X_{t_{GJ}}$ and $Y_{t_{GJ}}$ by taking only positive eigenvalues and replacing negative eigenvalues with zeros. The drawback of this technique is that it also have the non-similar effect by only affecting the negative eigenvalues. MI II [17] ensured the positive definiteness of the matrices $X_{t_{GJ}}$ and $Y_{t_{GJ}}$ by subtracting the smallest eigenvalue from all eigenvalues of the matrices $X_{t_{GJ}}$ and $Y_{t_{GJ}}$. The drawback of this technique is that last eigenvalue becomes zero which causes large error.

Proposed techniques overcome the issue of stability and large approximation error. It ensures the stability of the ROM and also gives least approximation error as compared to the existing TLMOR techniques.

3.1.1 Proposed Technique I

This Proposed technique I give stable ROM and produce less approximation error. In this technique the symmetric matrices $X_{t_{GJ}}$ and $Y_{t_{GJ}}$ are made positive definitive by subtracting the smallest negative eigenvalue of $X_{t_{GJ}}$ and $Y_{t_{GJ}}$ from S_{e_2} and R_{e_2} respectively. Let new controllability P_{S_1} and observability Q_{S_1} Gramians satisfying:

$$AP_{S_1} + P_{S_1}A^T + B_{S_1}B_{S_1}^T = 0 \tag{3.1}$$

$$A^TQ_{S_1} + Q_{S_1}A + C_{S_1}^TC_{S_1} = 0 \tag{3.2}$$

The updated input and output matrices B_{S_1} and C_{S_1} respectively are defined as;

$$B_{S_1} = \begin{cases} U_{S_1}\begin{bmatrix} S_{e_1} & 0 \\ 0 & S_{e_2} - s_n I_{(n-l)*(n-l)} \end{bmatrix}^{1/2} & \text{for } s_n < 0 \\ U_{S_1}(S_{S_1})^{1/2} & \text{for } s_n \geq 0 \end{cases} \tag{3.3}$$

$$C_{S_1} = \begin{cases} \begin{bmatrix} R_{e_1} & 0 \\ 0 & R_{e_2} - r_n I_{(n-k)*(n-k)} \end{bmatrix}^{1/2} V_{S_1}^T & \text{for } r_n < 0 \\ (R_{S_1})^{1/2}V_{S_1}^T & \text{for } r_n \geq 0. \end{cases} \tag{3.4}$$

The expressions U_{S_1}, S_{S_1}, R_{S_1} and V_{S_1} are determined by the SVD of the symmetric matrices $X_{t_{GJ}}$ and $Y_{t_{GJ}}$ as

$$X_{t_{GJ}} = U_{S_1}S_{S_1}U_{S_1}^T$$
$$Y_{t_{GJ}} = V_{S_1}R_{S_1}V_{S_1}^T$$

where

$$U_{S_1} = \begin{bmatrix} U_{e_1} & U_{e_2} \end{bmatrix}, S_{S_1} = \begin{bmatrix} S_{e_1} & 0 \\ 0 & S_{e_2} \end{bmatrix}$$

$$V_{S_1} = \begin{bmatrix} V_{e_1} & V_{e_2} \end{bmatrix}, R_{S_1} = \begin{bmatrix} R_{e_1} & 0 \\ 0 & R_{e_2} \end{bmatrix}$$

where

$$\begin{bmatrix} S_{e_1} & 0 \\ 0 & S_{e_2} \end{bmatrix} = \begin{bmatrix} s_{e_1} & 0 & \dots & 0 \\ 0 & s_{e_2} & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & s_{e_n} \end{bmatrix}$$

$$\begin{bmatrix} R_{e_1} & 0 \\ 0 & R_{e_2} \end{bmatrix} = \begin{bmatrix} r_{e_1} & 0 & \dots & 0 \\ 0 & r_{e_2} & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & r_{e_n} \end{bmatrix}$$

$$S_{e_1} = diag(s_{e_1}, s_{e_2}, \cdots, s_{e_l}), S_{e_2} = diag(s_{e_{l+1}}, s_{e_{l+2}}, \cdots, s_{e_n})$$

$$R_{e_1} = diag(r_{e_1}, r_{e_2}, \cdots, r_{e_l}), R_{e_2} = diag(r_{e_{l+1}}, r_{e_{l+2}}, \cdots, r_{e_n})$$

where $s_{e_1} \geq s_{e_2} \geq ... \geq s_{e_n}$ and $r_{e_1} \geq r_{e_2} \geq ... \geq r_{e_n}$ Let the contragradient transformation matrix T_{S_1} be obtained using the updated controllability and observability Gramians P_{S_1} and Q_{S_1} respectively.

$$T_{S_1}^T Q_{S_1} T_{S_1} = T_{S_1}^{-1} P_{S_1} T_{S_1}^{-T} = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & \sigma_n \end{bmatrix}$$

where $\sigma_j \geq \sigma_{j+1}, j = 1, 2, 3, ..., n-1$. Contragradient transformation T_{S_1} is applied to the original system to get an internally balanced realization.

$$\left[\begin{array}{c|c} A_{b_1} & B_{b_1} \\ \hline C_{b_1} & D_{b_1} \end{array}\right] = \left[\begin{array}{c|c} T_{S_1}^{-1} A T_{S_1} & T_{S_1}^{-1} B \\ \hline C T_{S_1} & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \tag{3.5}$$

ROM is achieved by segregating the least controllable and least observable states of the balanced realization.

$$G_{kk_1}(s) = C_1(sI - A_{11})^{-1} B_1 + D_1 \tag{3.6}$$

where $\{A_{11}, B_1, C_1, D_1\}$ is the k^{th} $(k < n)$ order minimal realization.

Remark 13 *Since* $X_{t_{GJ}} \leq B_{S_1}B_{S_1}^T \leq B_{t_{GA}}B_{t_{GA}}^T \geq 0$, $Y_{t_{GJ}} \leq C_{S_1}C_{S_1}^T \leq C_{t_{GA}}C_{t_{GA}}^T \geq 0$ *and the realization* $\{A, B_{S_1}, C_{S_1}\}$ *is minimal, therefore the ROM obtained is stable.*

Theorem 4 *For proposed technique I the following error bound formula holds provided that the following rank conditions* $rank\,[B_{S_1}\ \ B] = rank\,[B_{S_1}]$ *and* $rank \begin{bmatrix} C_{S_1} \\ C \end{bmatrix} = rank\,[C_{S_1}]$ *(which follows from [25]) are satisfied*

$$\|G(s) - G_{kk}(s)\|_\infty \leq 2\|L_{S_1}\|\|K_{S_1}\| \sum_{w=l+1}^{n} \sigma_w$$

$$L_{S_1} = \begin{cases} CV_{S_1}(R_{upd})^{-1/2} & \textit{for } r_n < 0 \\ CV_{S_1}(R_{S_1})^{-1/2} & \textit{for } r_n \geq 0 \end{cases} \tag{3.7}$$

$$K_{S_1} = \begin{cases} (S_{upd})^{-1/2}U_{S_1}^T B & \textit{for } s_n < 0 \\ (S_{S_1})^{-1/2}U_{S_1}^T B & \textit{for } s_n \geq 0. \end{cases} \tag{3.8}$$

where

$$R_{upd} = \begin{bmatrix} R_{e_1} & 0 \\ 0 & R_{e_2} - r_n I_{(n-k)*(n-k)} \end{bmatrix}, S_{upd} = \begin{bmatrix} S_{e_1} & 0 \\ 0 & S_{e_2} - s_n I_{(n-l)*(n-l)} \end{bmatrix}$$

Proof: Since $rank\,[B_{S_1}\ \ B] = rank\,[B_{S_1}]$ and $rank \begin{bmatrix} C_{S_1} \\ C \end{bmatrix} = rank\,[C_{S_1}]$, the relationships $B = B_{S_1}K_{S_1}$ and $C = L_{S_1}C_{S_1}$ hold. By partitioning $B_{S_1} = \begin{bmatrix} B_{e_1} \\ B_{e_2} \end{bmatrix}$, $C_{S_1} = \begin{bmatrix} C_{e_1} & C_{e_2} \end{bmatrix}$ and substituting $B_1 = B_{e_1}K_{S_1}$, $C_1 = L_{S_1}C_{e_1}$ respectively produces

$$\begin{aligned}
\|G(s) - G_{kk}(s)\|_\infty &= \|C(sI-A)^{-1}B - C_1(sI-A_{11})^{-1}B_1\|_\infty \\
&= \|L_{S_1}C_{S_1}(sI-A)^{-1}B_{S_1}K_{S_1} - L_{S_1}C_{e_1}(sI-A_{11})^{-1}B_{e_1}K_{S_1}\|_\infty \\
&= \|L_{S_1}(C_{S_1}(sI-A)^{-1}B_{S_1} - C_{e_1}(sI-A_{11})^{-1}B_{e_1})K_{S_1}\|_\infty \\
&\leq \|L_{S_1}\|\|(C_{S_1}(sI-A)^{-1}B_{S_1} - C_{e_1}(sI-A_{11})^{-1}B_{e_1})\|_\infty\|K_{S_1}\|
\end{aligned}$$

If $\{A_{11}, B_{e_1}, C_{e_1}\}$ is ROM achieved by segregating a balanced system realization $\{A, B_{S_1}, C_{S_1}\}$, we have from [2].

$$\|(C_{S_1}(sI-A)^{-1}B_{S_1} - C_{e_1}(sI-A_{11})^{-1}B_{e_1})\|_\infty \leq 2\sum_{w=l+1}^{n}\sigma_w.$$

Therefore,

$$\|G(s) - G_{kk}(s)\|_\infty \leq 2\|L_{S_1}\|\|K_{S_1}\| \sum_{w=l+1}^{n} \sigma_w$$

3.1.2 Proposed Technique II

This technique also produces stable ROM and also give less approximation error. In this technique the succeeding eigenvalue is subtracted from the prior eigenvalue of respective $X_{t_{GJ}}$ and $Y_{t_{GJ}}$ matrices. Let new controllability P_{S_2} and observability Q_{S_2} Gramians satisfying:

$$AP_{S_2} + P_{S_2}A^T + B_{S_2}B_{S_2}^T = 0 \tag{3.9}$$

$$A^TQ_{S_2} + Q_{S_2}A + C_{S_2}^TC_{S_2} = 0 \tag{3.10}$$

The updated input and output matrices B_{S_2} and C_{S_2} are defined as

$$B_{S_2} = \begin{cases} U_{S_2}(\hat{S_{S_2}})^{1/2} & \text{for } s_n < 0 \\ U_{S_2}(S_{S_2})^{1/2} & \text{for } s_n \geq 0 \end{cases} \tag{3.11}$$

$$C_{S_2} = \begin{cases} (\hat{R_{S_2}})^{1/2}V_{S_2}^T & \text{for } r_n < 0 \\ (R_{S_2})^{1/2}V_{S_2}^T & \text{for } r_n \geq 0. \end{cases} \tag{3.12}$$

where

$$\hat{S}_{S_2} = \begin{bmatrix} \hat{s}_{h_1} & 0 & \dots & 0 \\ 0 & \hat{s}_{h_2} & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & \hat{s}_{h_n} \end{bmatrix}, \hat{s}_{h_1} = s_{h_1}, \hat{s}_{h_{1+q}} = s_{g_{1+h-1}} - s_{h_{1+q}}$$

$$\hat{R}_{S_2} = \begin{bmatrix} \hat{r}_{h_1} & 0 & \dots & 0 \\ 0 & \hat{r}_{h_2} & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & \hat{r}_{h_n} \end{bmatrix}, \hat{r}_{h_1} = r_{h_1}, \hat{r}_{h_{1+u}} = r_{h_{1+u-1}} - r_{h_{1+u}}$$

$q = 1, 2, ..., n-1, u = 1, 2, ..., n-1$ The expressions U_{S_2}, S_{S_2}, R_{S_2} and V_{S_2} are determined by the SVD of the symmetric matrices $X_{t_{GJ}}$ and $Y_{t_{GJ}}$ as

$$\begin{aligned} X_{t_{GJ}} &= U_{S_2} S_{S_2} U_{S_2}^T \\ Y_{t_{GJ}} &= V_{S_2} R_{S_2} V_{S_2}^T \end{aligned}$$

where

$$U_{S_2} = \begin{bmatrix} U_{h_1} & U_{h_2} \end{bmatrix}, S_{S_2} = \begin{bmatrix} S_{h_1} & 0 \\ 0 & S_{h_2} \end{bmatrix}$$

$$V_{S_2} = \begin{bmatrix} V_{h_1} & V_{h_2} \end{bmatrix}, R_{S_2} = \begin{bmatrix} R_{h_1} & 0 \\ 0 & R_{h_2} \end{bmatrix}$$

where

$$\begin{aligned} \begin{bmatrix} S_{h_1} & 0 \\ 0 & S_{h_2} \end{bmatrix} &= \begin{bmatrix} s_{h_1} & 0 & \dots & 0 \\ 0 & s_{h_2} & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & s_{h_n} \end{bmatrix} \\ \begin{bmatrix} R_{h_1} & 0 \\ 0 & R_{h_2} \end{bmatrix} &= \begin{bmatrix} r_{h_1} & 0 & \dots & 0 \\ 0 & r_{h_2} & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & r_{h_n} \end{bmatrix} \end{aligned}$$

$$S_{h_1} = diag(s_{h_1}, s_{h_2}, \cdots, s_{h_l}), S_{h_2} = diag(s_{h_{l+1}}, s_{h_{l+2}}, \cdots, s_{h_n})$$

$$R_{h_1} = diag(r_{h_1}, r_{h_2}, \cdots, r_{h_l}), R_{h_2} = diag(r_{h_{l+1}}, r_{h_{l+2}}, \cdots, r_{h_n})$$

where $s_{h_1} \geq s_{h_2} \geq ... \geq s_{h_n}$ and $r_{h_l} \geq r_{h_2} \geq ... \geq r_{h_n}$ Let the contragradient transformation matrix T_{S_2} be obtained using the new controllability and observability Gramians P_{S_2} and Q_{S_2} respectively.

$$T_{S_2}^T Q_{S_2} T_{S_2} = T_{S_2}^{-1} P_{S_2} T_{S_2}^{-T} = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & \sigma_n \end{bmatrix}$$

where $\sigma_j \geq \sigma_{j+1}, j = 1, 2, 3, ..., n-1$. Contragradient transformation T_{S_2} is applied to the original system to get an internally balanced realization.

$$\left[\begin{array}{c|c} A_{b_2} & B_{b_2} \\ \hline C_{b_2} & D_{b_2} \end{array}\right] = \left[\begin{array}{c|c} T_{S_2}^{-1} A T_{S_2} & T_{S_2}^{-1} B \\ \hline C T_{S_2} & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \tag{3.13}$$

ROM is achieved by segregating the least controllable and least observable states of the balanced realization.

$$G_{kk_2}(s) \quad = \quad C_1(sI - A_{11})^{-1} B_1 + D_1 \tag{3.14}$$

where $\{A_{11}, B_1, C_1, D_1\}$ is the k^{th} $(k < n)$ order minimal realization.

Remark 14 *Since $X_{t_{GJ}} \leq B_{S_2} B_{S_2}^T \leq B_{t_{GA}} B_{t_{GA}}^T \geq 0$, $X_{t_{GJ}} \leq C_{S_2} C_{S_2}^T \leq C_{t_{GA}} C_{t_{GA}}^T \geq 0$ and the realization $\{A, B_{S_2}, C_{S_2}\}$ is minimal, therefore the ROM obtained is stable.*

Theorem 5 *For proposed technique II the following error bound formula holds provided that the following rank conditions $rank\,[B_{S_2} \;\; B] = rank\,[B_{S_2}]$ and $rank \begin{bmatrix} C_{S_2} \\ C \end{bmatrix} = rank\,[C_{S_2}]$ (which follows from [25]) are satisfied*

$$\|G(s) - G_{kk}(s)\|_\infty \leq 2\|L_{S_2}\|\|K_{S_2}\| \sum_{w=l+1}^{n} \sigma_w$$

$$L_{S_2} = \begin{cases} CV_{S_2}(\hat{R}_{S_2})^{-1/2} & \textit{for } r_n < 0 \\ CV_{S_2}(R_{S_2})^{-1/2} & \textit{for } r_n \geq 0 \end{cases} \tag{3.15}$$

$$K_{S_2} = \begin{cases} (\hat{S}_{S_2})^{-1/2}U_{S_2}^T B & \textit{for } s_n < 0 \\ (S_{S_2})^{-1/2}U_{S_2}^T B & \textit{for } s_n \geq 0. \end{cases} \tag{3.16}$$

$$\hat{S}_{S_2} = \begin{bmatrix} \hat{s}_{h_1} & 0 & \dots & 0 \\ 0 & \hat{s}_{h_2} & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & \hat{s}_{h_n} \end{bmatrix}, \hat{s}_{h_1} = s_{h_1}, \hat{s}_{h_{1+q}} = s_{h_{1+q-1}} - s_{h_{1+q}}$$

$$\hat{R}_{S_2} = \begin{bmatrix} \hat{r}_{h_1} & 0 & \dots & 0 \\ 0 & \hat{r}_{h_2} & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & \hat{r}_{h_n} \end{bmatrix}, \hat{r}_{h_1} = r_{h_1}, \hat{r}_{h_{1+u}} = r_{h_{1+u-1}} - r_{h_{1+u}}$$

$q = 1, 2, ..., n-1, u = 1, 2, ..., n-1$

Proof: Since $rank\,[B_{S_2} \;\; B] = rank\,[B_{S_2}]$ and $rank \begin{bmatrix} C_{S_2} \\ C \end{bmatrix} = rank\,[C_{S_2}]$, the relationships $B = B_{S_2}K_{S_2}$ and $C = L_{S_2}C_{S_2}$ hold. By partitioning $B_{S_2} = \begin{bmatrix} B_{h_1} \\ B_{h_2} \end{bmatrix}, C_{S_2} = \begin{bmatrix} C_{h_1} & C_{h_2} \end{bmatrix}$ and substituting $B_1 = B_{h_1}K_{S_2}, C_1 = L_{S_2}C_{h_1}$ respectively produces

$$\begin{aligned} \|G(s) - G_{kk}(s)\|_\infty &= \|C(sI-A)^{-1}B - C_1(sI-A_{11})^{-1}B_1\|_\infty \\ &= \|L_{S_2}C_{S_2}(sI-A)^{-1}B_{S_2}K_{S_2} - L_{S_2}C_{h_1}(sI-A_{11})^{-1}B_{h_1}K_{S_2}\|_\infty \\ &= \|L_{S_2}(C_{S_2}(sI-A)^{-1}B_{S_2} - C_{h_1}(sI-A_{11})^{-1}B_{h_1})K_{S_2}\|_\infty \\ &\leq \|L_{S_2}\|\|(C_{S_2}(sI-A)^{-1}B_{S_2} - C_{h_1}(sI-A_{11})^{-1}B_{h_1})\|_\infty\|K_{S_2}\| \end{aligned}$$

If $\{A_{11}, B_{h_1}, C_{h_1}\}$ is ROM achieved by segregating a balanced system realization $\{A, B_{S_2}, C_{S_2}\}$, we have from [2].

$$\|(C_{S_2}(sI-A)^{-1}B_{S_2} - C_{h_1}(sI-A_{11})^{-1}B_{h_1})\|_\infty \leq 2\sum_{w=l+1}^{n}\sigma_w.$$

Therefore,

$$\|G(s) - G_{kk}(s)\|_\infty \leq 2\|L_{S_2}\|\|K_{S_2}\| \sum_{w=l+1}^{n} \sigma_w$$

Chapter 4

Results and Discussion

In this chapter some numerical examples are presented to show the effectiveness of proposed techniques when compared with existing TLMOR techniques.

4.1 Numerical Examples

Example 1: Consider a linear time invariant stable 6^{th} order system with the following state space representation

$$A=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -5.45 & 4.54 & 0 & -0.05 & 0.04 & 0 \\ 10 & -21 & 11 & 0.1 & -0.21 & 0.11 \\ 0 & 5.5 & -6.5 & 0 & 0.05 & -0.06 \end{bmatrix}, B=\begin{bmatrix} 0 \\ 00 \\ 0 \\ 0.09 \\ 0.4 \\ -0.5 \end{bmatrix}$$

$$C=\begin{bmatrix} 2 & -2 & 3 & 0 & 0 & 0 \end{bmatrix}, D=0$$

The 1^{st} order ROM obtained by GJ [11] is unstable with pole $s = 0.0413$ while the ROM obtained by proposed techniques are stable within the desired time interval $[t_1, t_2] = [0, 7]$ sec as shown in Table no 4.1.

Example 2: Consider a linear time invariant (LTI) stable system of order 8 with the following

Table 4.1: Poles location of the reduced order systems

Techniques	Pole location
GJ [11]	0.0413
GA [12]	-0.0032
MI I [17]	-0.0039
MI II [17]	-0.0040
Proposed Technique I	-0.0034
Proposed Technique II	-0.0033

state space representation.

$$
\begin{aligned}
A &= \begin{bmatrix}
-10.23 & -10.19 & 0 & 0 & 3.46 & 0 & 0 & 0 \\
10.19 & 0 & 0 & 0 & 0 & 3.46 & 0 & 0 \\
-10.23 & 28.04 & -0.83 & -11.15 & 0 & 0 & 3.46 & 0 \\
0 & 0 & 11.15 & 0 & 0 & 0 & 0 & 3.46 \\
-3.46 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3.46 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3.46 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -3.46 & 0 & 0 & 0 & 0
\end{bmatrix} \\
B &= \begin{bmatrix} 11 & 0 & 11 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\
C &= \begin{bmatrix} -0.20 & 0.57 & -0.01 & 0.02 & 0 & 0 & 0 & 0 \end{bmatrix} \\
D &= 0
\end{aligned}
$$

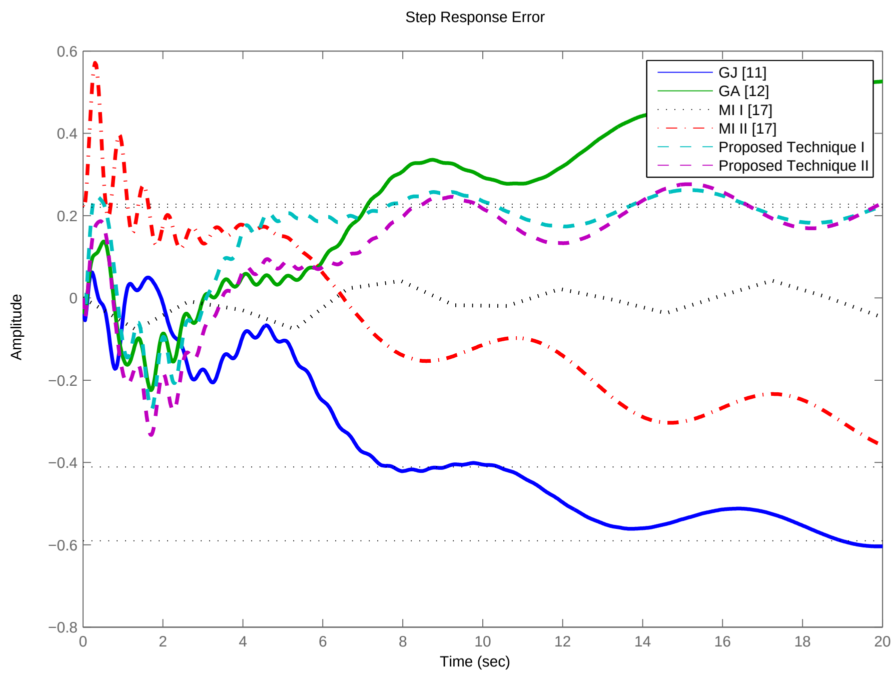

Figure 4.1: Step Response error

Fig 4.1 illustrates the plot of the response error of the 5^{th} order ROM obtained by the techniques GJ [11], GA [12], MI I [17], MI II [17], Proposed technique I and Proposed technique II in the desired time interval $[t_1, t_2] = [0, 20]$sec. Fig 4.1 shows that both proposed techniques give least approximation error as compared to the existing techniques GJ [11], GA [12], MI I [17] and MI II [17].

Fig 4.2 illustrates the plot of the impulse response error of the of the 5^{th} order ROM

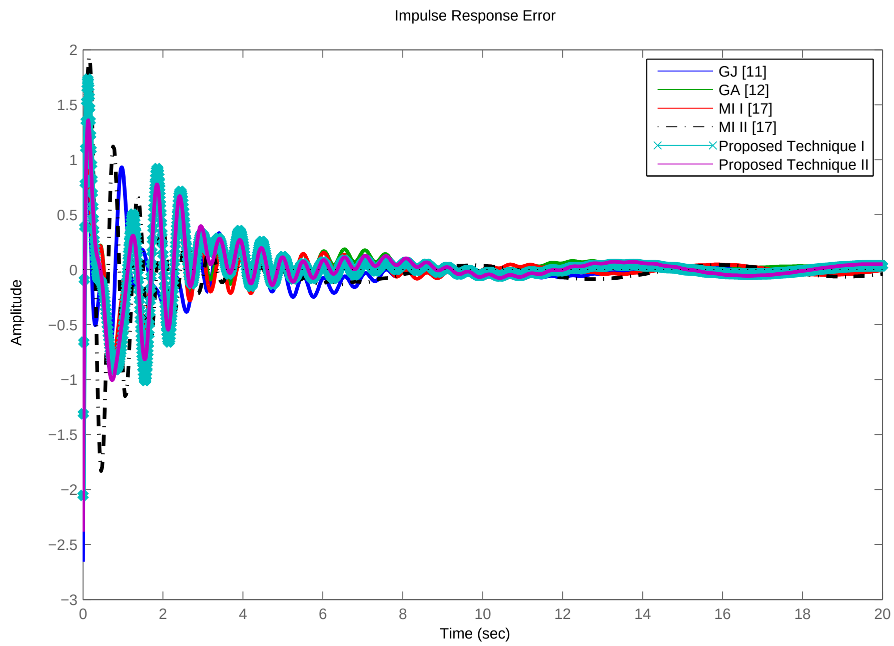

Figure 4.2: Impulse Response error

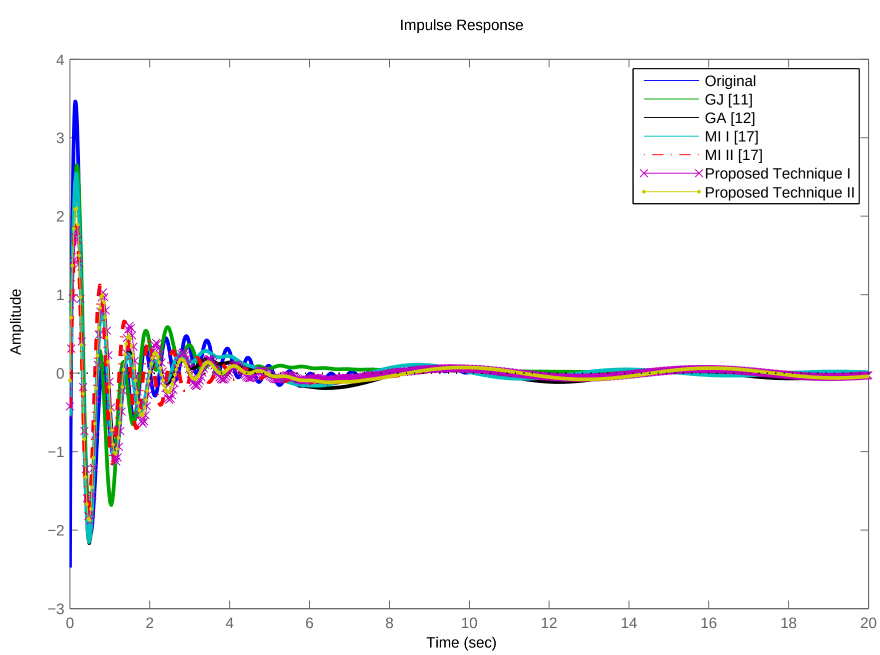

Figure 4.3: Impulse response

obtained by the techniques GJ [11], GA [12], MI I [17], MI II [17], Proposed technique I and Proposed technique II in the desired time interval $[t_1, t_2] = [0, 20]$sec. Fig 4.3 illustrates the plot of the impulse response of the original system and impulse response of the 5^{th} order ROM obtained by the techniques GJ [11], GA [12], MI I [17], MI II [17], Proposed technique I and Proposed technique II in the desired time interval $[t_1, t_2] = [0, 20]$sec. It is observed from Fig 4.1-4.3 that proposed techniques yield less step and impulse response error when compared with existing TLMOR techniques.

Example 3: Consider a LTI stable system of order 12 with the following state space representation.

$$A = \begin{bmatrix} -1.26 & -3.17 & 0 & 0 & 0 & 0 & 6.70 & 0 & 0 & 0 & 0 & 0 \\ 3.17 & 0 & 0 & 0 & 0 & 0 & 0 & 6.70 & 0 & 0 & 0 & 0 \\ 0 & 45.36 & -0.92 & -8.51 & 0 & 0 & 0 & 0 & 6.70 & 0 & 0 & 0 \\ 0 & 0 & 8.51 & 0 & 0 & 0 & 0 & 0 & 0 & 6.70 & 0 & 0 \\ 0 & 0 & 0 & 16.92 & -0.33 & -11.60 & 0 & 0 & 0 & 0 & 6.70 & 0 \\ 0 & 0 & 0 & 0 & 11.60 & 0 & 0 & 0 & 0 & 0 & 0 & 6.70 \\ -6.70 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6.70 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6.70 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6.70 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6.70 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6.70 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.01 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D = 0$$

Fig 4.4 illustrates the plot of the response error of the 6^{th} order ROM obtained by the techniques GJ [11], GA [12],MI I [17], MI II [17], Proposed technique I and Proposed technique II in the desired time interval $[t_1, t_2] = [10, 27]$sec. Fig 4.4 shows that both the

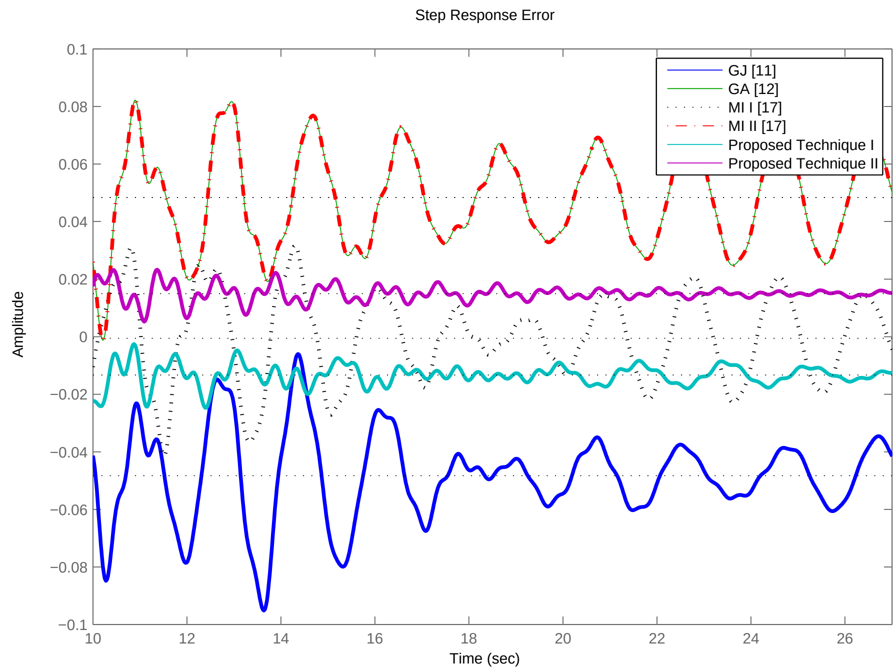

Figure 4.4: Step Response error

proposed techniques give least approximation error as compared to the existing techniques GJ [11], GA [12], MI I [17] and MI II [17].

Fig 4.5 illustrates the plot of the impulse response error of the of the 6^{th} order ROM obtained by the techniques GJ [11], GA [12], MI I [17], MI II [17], Proposed technique I and Proposed technique II in the desired time interval $[t_1, t_2] = [10, 27]$sec. Fig 4.6 illustrates the plot of the impulse response of the original system and impulse response of the 6^{th} order ROM obtained by the techniques GJ [11], GA [12], MI I [17], MI II [17], Proposed technique I and Proposed technique II in the desired time interval $[t_1, t_2] = [10, 27]$sec. It is observed from Fig 4.4-4.6 that proposed techniques yield less step and impulse response error when compared with existing TLMOR techniques.

Example 4: Consider an analogue chebyshev type 1 bandpass filter of 30th order with pass-

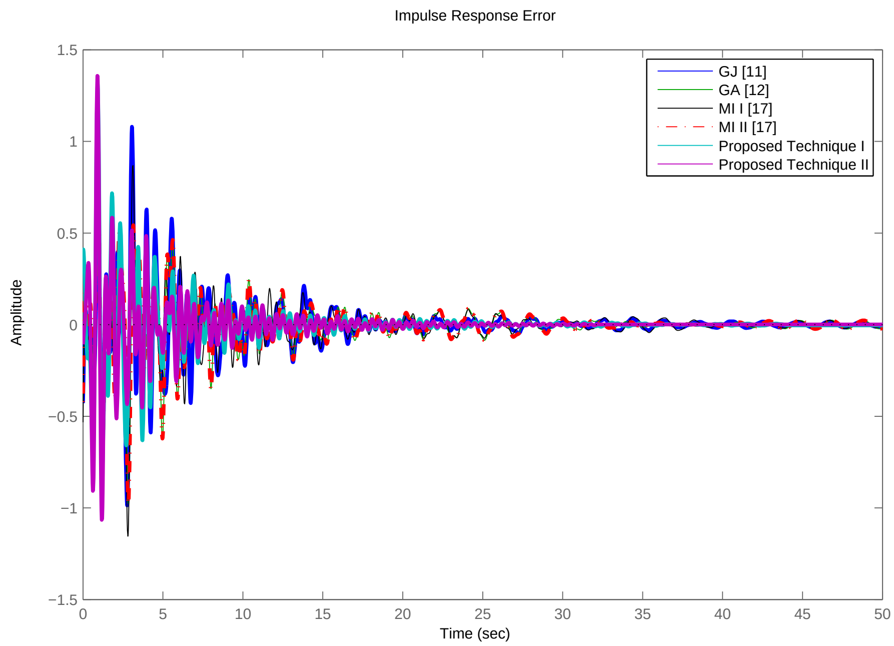

Figure 4.5: Impulse Response error

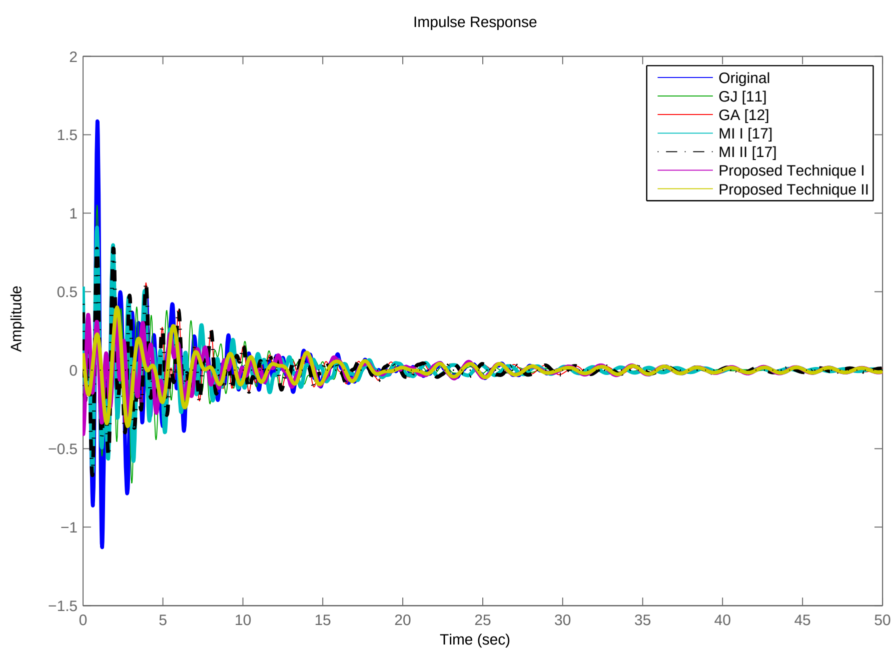

Figure 4.6: Impulse response

band ripple of 7 dB.

$$G(s)=\frac{\begin{array}{r}-1.155e^{-14}s^{29}-4.093e^{-12}s^{28}-5.639e^{-11}s^{27}-9.313e^{-9}s^{26}-\\ 9.127e^{-8}s^{25}-1.24e^{-5}s^{24}-8.965e^{-5}s^{23}-0.01074s^{22}-0.06152s^{21}-\\ 6.5s^{20}-29.5s^{19}-2496s^{18}-10240s^{17}-802816s^{16}+3.047e^{10}s^{15}-\\ 1.887e^{8}s^{14}-5.033e^{8}s^{13}-3.168e^{10}s^{12}-7.14e^{10}s^{11}-4.123e^{12}s^{10}-\\ 7.49e^{12}s^{9}-3.958e^{14}s^{8}-5.629e^{14}s^{7}-2.702e^{16}s^{6}-2.815e^{16}s^{5}-\\ 1.279e^{18}s^{4}-8.332e^{17}s^{3}-3.632e^{19}s^{2}-1.146e^{19}s-4.658e^{20}\end{array}}{\begin{array}{r}s^{30}+3.065s^{29}+3380s^{28}+9635s^{27}+5.245e^{6}s^{26}+1.382e^{7}s^{25}+4.956e^{9}s^{24}+1.2e^{10}s^{23}+\\ 3.188e12s^{22}+7.031e12s^{21}+1.478e15s^{20}+2.944e15s^{19}+5.103e17s^{18}+9.076e^{17}s^{17}+\\ 1.336e^{20}s^{16}+2.094e^{20}s^{15}+2.671e^{22}s^{14}+3.631e^{22}s^{13}+4.083e^{24}s^{12}+4.71e^{24}s^{11}+\\ 4.73e^{26}s^{10}+4.5e^{26}s^{9}+4.081e^{28}s^{8}+3.071e^{28}s^{7}+2.538e^{30}s^{6}+1.416e^{30}s^{5}+\\ 1.074e^{32}s^{4}+3.946e^{31}s^{3}+2.769e^{33}s^{2}+5.022e^{32}s+3.277e^{34}\end{array}}$$

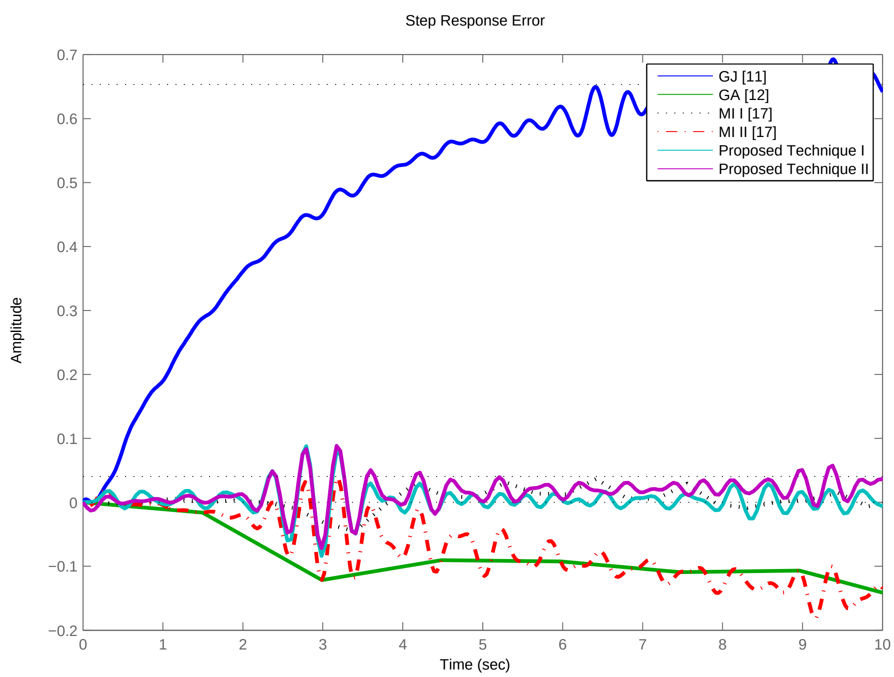

Figure 4.7: Step Response error

Fig 4.7 illustrates the plot of the response error of the 15^{th} order ROM obtained by the techniques GJ [11], GA [12], MI I [17], MI II [17], Proposed technique I and Proposed technique II in the desired time interval $[t_1, t_2] = [0, 10]$sec. Fig 4.7 shows that both the proposed techniques give least approximation error as compared to the existing techniques GJ [11], GA [12], MI I [17] and MI II [17].

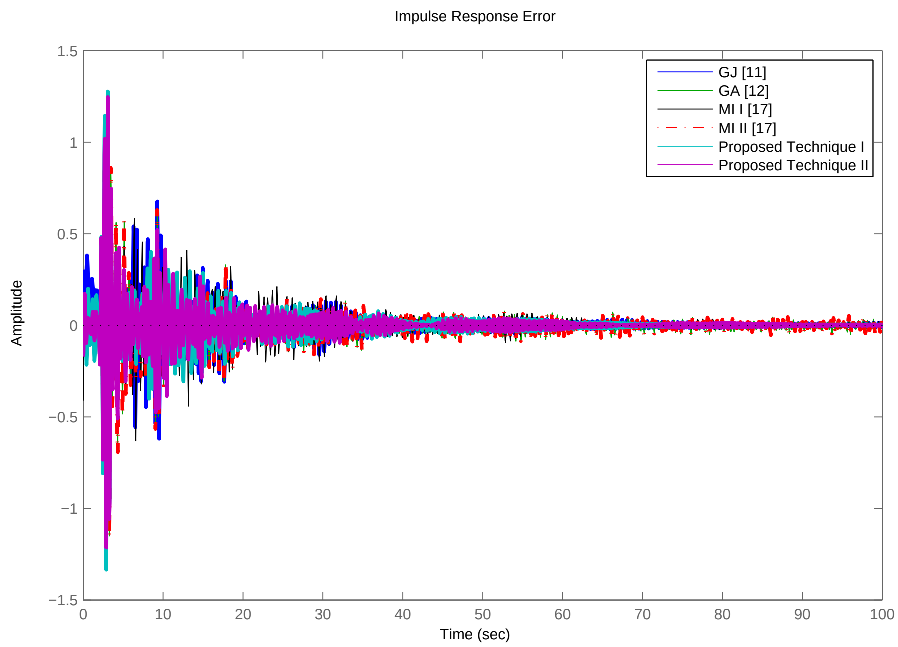

Figure 4.8: Impulse Response error

Fig 4.8 illustrates the plot of the impulse response error of the of the 15^{th} order ROM obtained by the techniques GJ [11], GA [12], MI I [17], MI II [17], Proposed technique I and Proposed technique II in the desired time interval $[t_1, t_2] = [0, 10]$sec.

Fig 4.9 illustrates the plot of the impulse response of the original system and impulse response of the 15^{th} order ROM obtained by the techniques GJ [11], GA [12], MI I [17], MI II [17], Proposed technique I and Proposed technique II in the desired time interval $[t_1, t_2] = [0, 10]$sec. It is observed from Fig 4.7-4.9 that proposed techniques yield less step and impulse response error when compared with existing TLMOR techniques.

Example 5:Consider an analogue chebyshev type 1 bandpass filter of 16^{th} order with pass-

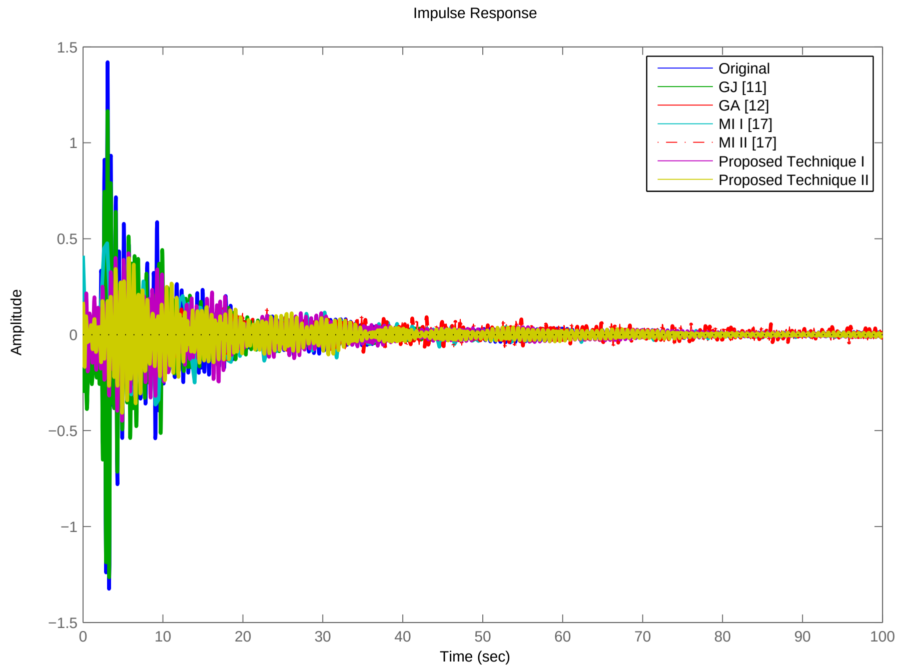

Figure 4.9: Impulse response

band ripple of 20 dB.

$$G(s)=\frac{\begin{array}{r}2.22e^{-15}s^{15}-2.274e^{-13}s^{14}+1.137e^{-12}s^{13}-1.455e^{-10}s^{12}+1.746e^{-10}s^{11}-\\2.235e^{-8}s^{10}+1.024e^{-8}s^{9}+3.376e^{5}s^{8}+6.557e^{-7}s^{7}-0.0001221s^{6}+\\8.583e^{-6}s^{5}-0.002686s^{4}+6.104e^{-5}s^{3}-0.02637s^{2}+0.0003052s-0.09961\end{array}}{\begin{array}{r}s^{16}+0.7715s^{15}+648.3s^{14}+427.7s^{13}+1.605e^{5}s^{12}+\\8.686e^{4}s^{11}+1.928e^{7}s^{10}+8.094e^{6}s^{9}+1.199e9s^{8}+3.642e8s^{7}+\\3.905e10s^{6}+7.915e9s^{5}+6.582e11s^{4}+7.893e^{10}s^{3}+5.383e^{12}s^{2}+2.883e^{11}s\end{array}}$$

Fig 4.10 and Fig 4.11 illustrates the unmagnified and magnified veiw respectively of plot of the response error of the 5^{th} order ROM obtained by the techniques GJ [11], GA [12], MI I [17], MI II [17], Proposed technique I and Proposed technique II in the desired time interval $[t_1, t_2] = [0, 16]$sec. Fig 4.10 and Fig 4.11 shows that both proposed techniques give least approximation error as compared to the existing techniques GJ [11], GA [12], MI I [17] and MI II [17].

Fig 4.12 illustrates the plot of the impulse response error of the of the 5^{th} order ROM obtained by the techniques GJ [11], GA [12], MI I [17], MI II [17], Proposed technique I

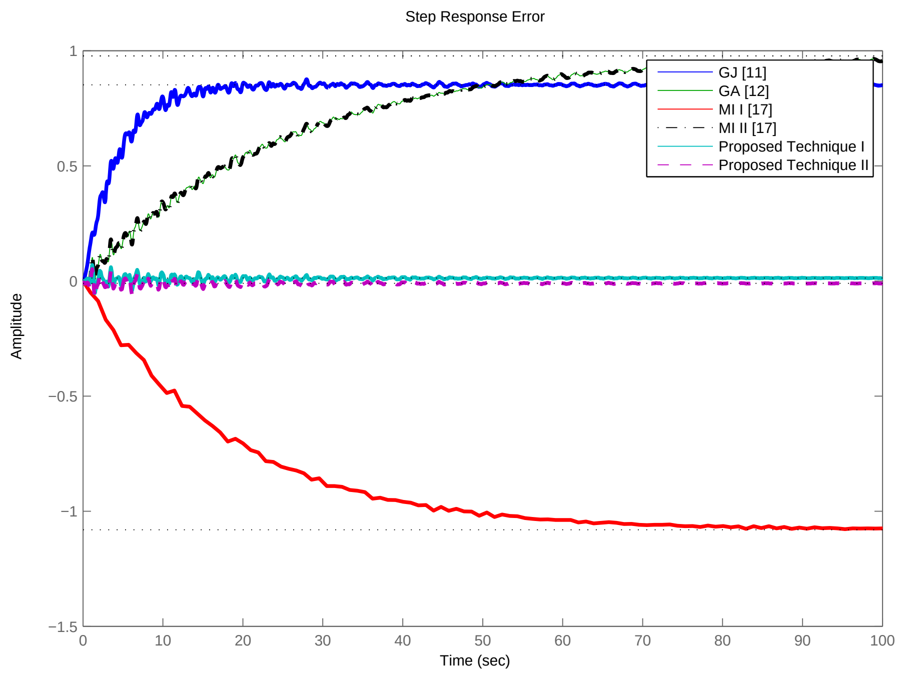

Figure 4.10: Step Response error

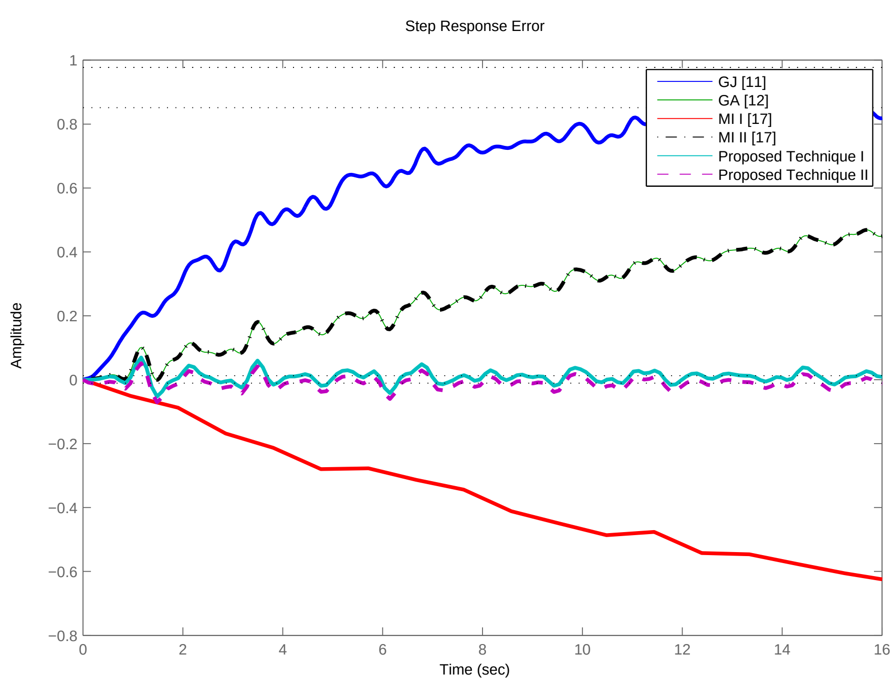

Figure 4.11: Zoom view of Step Response error

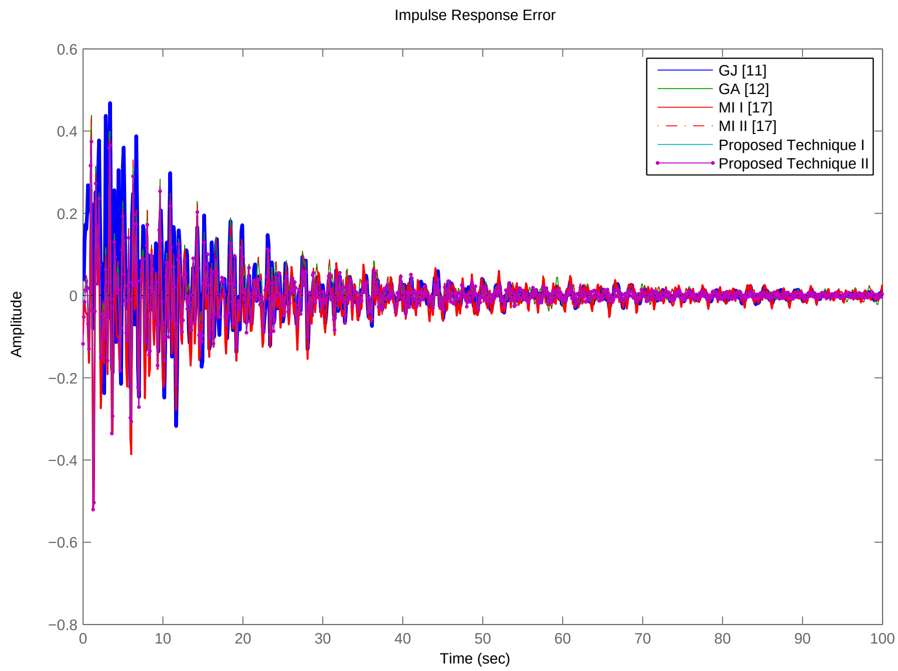

Figure 4.12: Impulse Response error

and Proposed technique II in the desired time interval $[t_1, t_2] = [0, 16]$sec.

Fig 4.13 illustrates the plot of the impulse response of the original system and impulse response of the 5^{th} order ROM obtained by the techniques GJ [11], GA [12], MI I [17], MI II [17], Proposed technique I and Proposed technique II in the desired time interval $[t_1, t_2] = [0, 16]$sec. It is observed from Fig 4.10-4.13 that proposed techniques yield less step and impulse response error when compared with existing TLMOR techniques.

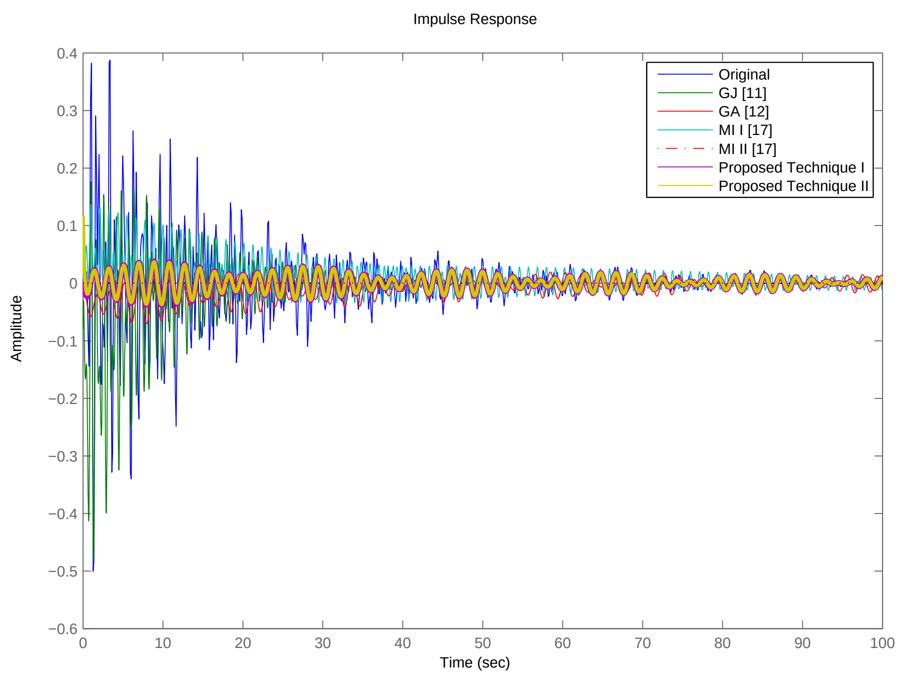

Figure 4.13: Impulse response

Chapter 5

Conclusion

5.1 Conclusion

In this thesis two new TLMOR techniques have been proposed that not only preserve stability in ROMs but also carries error bounds. Moreover, proposed techniques yield least approximation error as compared to existing TLMOR techniques. It is observed from numerical results that GJ [11] TLMOR technique yield less approximation error but it may sometime yield unstable ROM. Proposed techniques yield less approximation error as compared to existing stability preserving TLMOR techniques.

5.2 Future Directions

A lots of work has been done in FWMOR and FLMOR but in TLMOR there are still many open areas for research. Some improvements are needed in this area, that are given below:

- Existing techniques like wang et al's, VA [6], GA [12], GS [4] and IG [13] and proposed techniques are realization dependant, where original system realization produces lower approximation error and tight error bounds needs attention.
- Existing techniques may be extended and applied in time weighted MOR.
- TLMOR techniques are not relavent for non-linear systems. So, in future TLMOR techniques with some suitable improvisations may be applied in non-linear systems.
- In this thesis first order systems are used. In future TLMOR techniques could be applied in second order systems.
- TLMOR techniques use BT [1] . Different other MOR techniques like Krylov, Hankel norm, Pade approximation techniques could be used in future.
- Stability of the ROM in GJ's FLMOR and TLMOR technique is not ensured. This will remain an open area for future research.

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