STABILITY PRESEVING FREQUENCY LIMITED MODEL REDUCTION TECHNIQUES

By

Hira Zahid

A thesis submitted to the faculty of Electrical Engineering Department, Military College of Signals, National University of Sciences and Technology, Islamabad, Pakistan, in partial fulfillment of the requirements for the degree of MS in Electrical (Telecommunication) Engineering

APRIL 2020

THESIS ACCEPTANCE CERTIFICATE

Certified that final copy of MS/MPhil thesis written by MS **Hira Zahid,** Registration No. **00000172299** of **Military College of Signals** has been vetted by undersigned, found complete in all respect as per NUST Statutes/Regulations, is free of plagiarism, errors and mistakes and is accepted as partial, fulfillment for award of MS/MPhil degree. It is further certified that necessary amendments as pointed out by GEC members of the student have been also incorporated in the said thesis.

Signature:_________________________

Name of Supervisor :Asst Prof Dr M Imran, PhD

Date:_____________________________

Signature (HoD):_____________________

Date:_____________________________

Signature (Dean):___________________

Date:_____________________________

Abstract

Model order reduction is the very challenging field of control system because the order of higher order system is reduce by using MOR. The mathematical models of high order dynamic systems can be described in different form either in state space or transfer function. These are expressed in time and frequency domain respectively. It is generally recommended for reducing the system order by maintaining the dominant properties of the original system. It will promote to make better understanding of physical system, computational and hardware complexities reduces and simplify the controller design. Ample amount of research have been done on model order reduction. Some existing methods reducing the full order system into a lesser order for a entire frequency ranges. However, there are some applications like controller and filter etc that require reduction over specific frequency band. That gives the basic for the using frequency weights in model order reduction. Moreover, prior frequency limited techniques have drawbacks of lacking properties such as stability, error bound and large approximation error. This thesis will focus on frequency limited model reduction problem. Firstly, problem of frequency limited MOR will be formulated and then novel frequency limited balanced MOR methods are purposed. These methods will yield stable reduced order models. The new measures will guarantee stability by specifying some fictitious semi positive/ positive definiteness of input and output related matrices. Each input and output matrices preserved positive definiteness of the

matrices, respectively, by defining new controllability Gramian and observability Gramian in a novel way. That guides towards a new transformation matrix which subsequently, results in stability preserving methods including computable error bounds and has a low approximation error.

Acknowledgment

Above all, I am very grateful to Allah Almighty who gave the courage and strength to me for pursue this research.

Pursuing a research for me is not an easy job and I would like to share my sincere gratitude to supervisor Dr. Muhammad Imran for his countless help or support,motivation, immense knowledge and patience throughout my research work. His guidance keep me focused and helped a lot in a whole research work and in thesis writing.

Besides my supervisor, I would particularly like to acknowledge to all my committee members Dr. Adil Masood Siddiqui, Dr. Safia Akram, Dr.Abdul Ghafoor. I am grateful to Ma'am Sammana Batool and Sir Imran; who guided me throughout my thesis. I would have not succeeded without their invaluable support and help.

Most importantly, the support and assistance of my family are commendable. I dedicate this thesis to my lovely son, whose smile is great source of encouragement and appreciation for me in my entire research.

Contents

List of Abbreviations

dB	Decibel
BT	Balance Truncation
MOR	Model Order Reduction
ROM	Reduced Order Model
EVD	Eigenvalue Decomposition
SVD	Singular Value Decomposition
HSV	Hankel Singular Values
diag	Diagonal
FWMOR	Frequency Weighted Model Order Reduction
FLMOR	Frequency Limited Model Order Reduction

List of Figures

List of Tables

Chapter 1

INTRODUCTION

1.1 Model Order Reduction

The dynamic behavior of a physical system is defined by an appropriate derivation of mathematical model and obtain the desired performance specifications of those systems. The modeling of complex systems provide a large scale systems. Although, with the improvement in an advance technology and the ever growing computation speed, analysis, chip design, optimization and control of wide scale systems is complicated because of computations and memory storage requirements. Hence, procedure of creating a low-dimensional or reduce order model (ROMs) that give better understanding of full order original system that called model order reduction (MOR). Usally, the main MOR objective is to figure out ROMs that approximate, the input and output behaviour of actual systems. ROM are acquired with lesser memory capacity requirements along with evaluation time. MOR has played prominent part in design of advance control system and gain a too much consideration in the recent decades [1] - [20]. The reduction error of MOR is most significant factor that is achieved from the difference between original and ROMs. Moreover, systems features such as stability, input and output performance and passivity are also evenly essential to be ensured in MOR. In MOR, the computational efficiency is improved by numerical characteristics of the systems like computation speed, accuracy and memory storage capacity. The error bound formula provides some vision of approximation error in MOR methods. That will facilitate the engineerer to decide MOR method for relevant applications. Figure 1.1 shows the basic process of MOR.

1.1.1 The Balanced Truncation

Let consider continuous linear time invariant system be

$$G(s) = C(sI - A)^{-1}B + D \tag{1.1}$$

in which, its n^{th} order minimal realization is $\{A, B, C, D\}$.Purpose of using MOR is to get ROM

$$G_r(s) = C_r(sI - A_r)^{-1}B_r + D, \tag{1.2}$$

where r^{th} order minimal realization ($r < n$) is A_{11}, B_1, C_1, D so that approximation error is minimal $\|G(s) - G_r(s)\|_\infty$. In MOR method, Balance truncation (BT) [1] is frequently used that ensure ROMs stability and give a existing frequency response error bounds. In BT, within an internally balanced system the controllability and observability Gramians are reformed. The controllable and observable states are same for internally balanced realization. The lowest controllable and observable states are truncated for acquired ROM. Thus, the error acquired is significantly lower, which represents better performance of ROMs by using BT [1] technique. Moreover BT [1], other schemes like Hankel optimal approximation [2], Krylov technique [23] and Pade approximation [24] etc. play prominent part in resolving problem of MOR. Balanced singular perturbation approximation (BSPA)[7] [8] is used for better performance at lower frequencies because, BT [1]

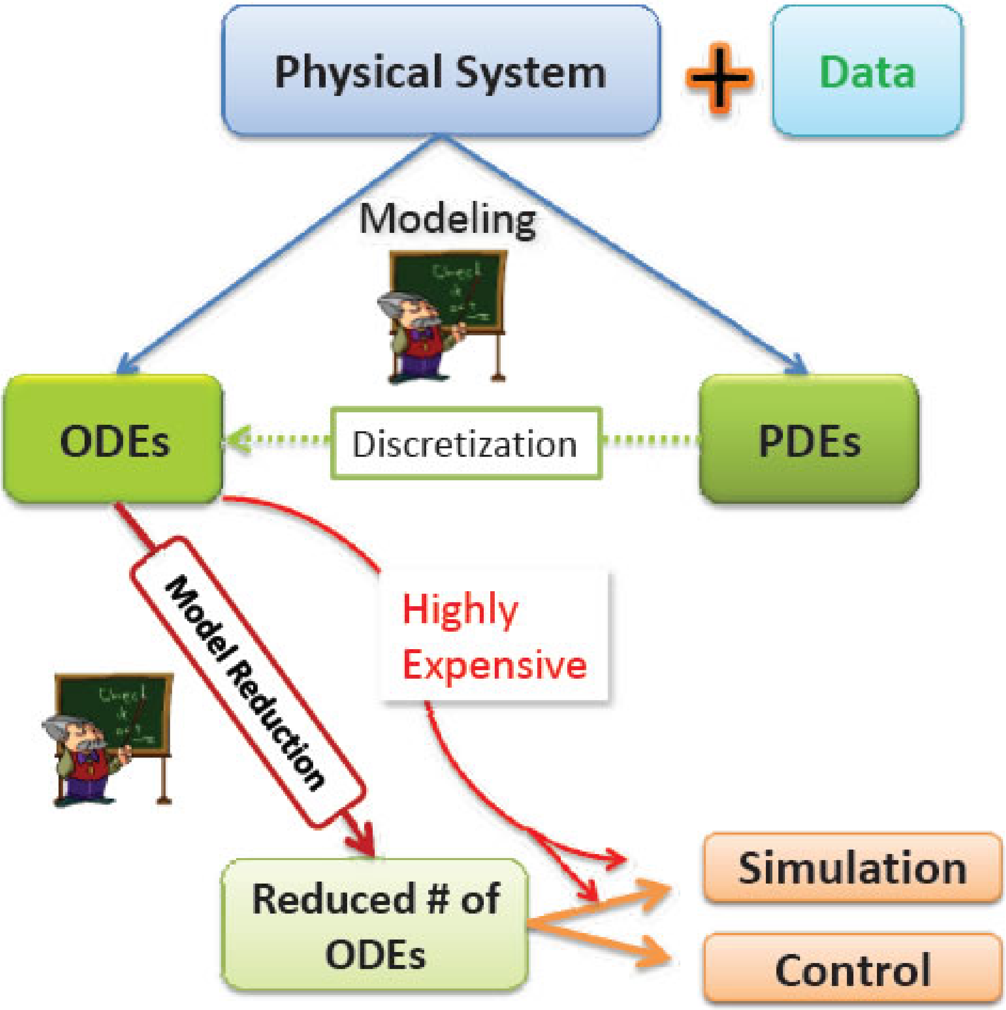

Figure 1.1: Model order reduction

performed better at higher frequencies. The ROMs acquired through the BSPA [7] [8] are also balanced and stable. Furthermore, the error bound also maintain for both the methods BT [1] and for BSPA [7] [8]. Nonetheless, even for precisely appropriate original systems, ROMs acquired by BSPA [7] [8] might be appropriate.

1.2 Frequency Weighting Model Order Reduction

It is vital that reduction error (reductio error is between original system and ROM) is smaller for all frequencies in MOR. In some cases, the reduction error is most importantly in a specify band of frequency as compare to other frequencies. That is the case, when ROMs are used in feedback control systems [3] [46]. It lead towards the idea of including frequency weighting in the process of MOR , is called the problem of frequency weighting model order reduction (FWMOR). Consider stable original system

$$G(s) = C(sI - A)^{-1}B + D, \tag{1.3}$$

the stable input/output weighting system

$$V_i(s) = C_v(sI - A_v)^{-1}B_v + D_v, \tag{1.4}$$

$$W_o(s) = C_w(sI - A_w)^{-1}B_w + D_w, \tag{1.5}$$

where A, B, C, D A_v, B_v, C_v, D_v A_w, B_w, C_w, D_w is its n^{th}, p^{th} and q^{th} minimal order realization respectively, the main purpose of MOR is to get a ROM

$$G_r(s) = C_r(sI - A_r)^{-1}B_r + D, \tag{1.6}$$

where A_r, B_r, C_r, D_r is an r^{th} order minimal realization ($r < n$) so that the weighting error $\| W_o(s)G(s) - G_r(s)V_i(s)\|_\infty$ become as small as possible. That is called as doubled sided FWMOR problem. If only input weights are used in system, this system is consider as one sided weighting system, it's aim is to get a ROM for the system, which is called input stable weighting $\|G(s) - G_r(s)V_i(s)\|_\infty$ and that is called output stable weighting $\|W_o(s)G(s) - G_r(s)\|\infty$ is make as low as possible. Enns [3] was first to incorporate frequency weights to BT [1] technique. Enns [3] technique might be use for input, output or both sided weightings. Whereas, stability of ROM is preserved for single sided weighting case but in case of double sided weighing stability of ROM has not ensured. To handle the instability issue in double sided weighting, various modification have been suggested in Enns approach [25], [4], [33], [34], [36], [5], [39], [46], [41], [42], [43], [44], [45]. In Lin and Chiu [4] method, ensured stability in double sided weights case that is drawback of Enns [3] method. Later, Sreeram et al [33] and Vargr and Anderson [34] introduced modification in Lin and Chiu [4] method due to its limitations. The Limitations are that when augmented system is formed then there is no occurrence of zero and pole cancellation. That technique work only well when proper weights is used. Wang et al. [5] method also provided solution for the Enns [3] instability issue along with yield stable ROM and error bound in case of double sided weightings. Later, approximation error of wang et al. [5] methodwas modified by Varga and Anderson [34] and Ghafoor and Sreeram [26]. Figure 1.2 show the double sided frequency weighting problem.

1.3 Frequency Limited Model Order Reduction

The FWMOR method is most frequently used to obtain reduction error in weighting system$\|W_o(s)G(s) - G_r(s)V_i(s)\|\infty$ as low as possible, where $V_i(s)$ is input and $W_o(s)$ is output weighting of the system [3]. Usually, the input/output weighting are fictitious because result may change by introducing any change the

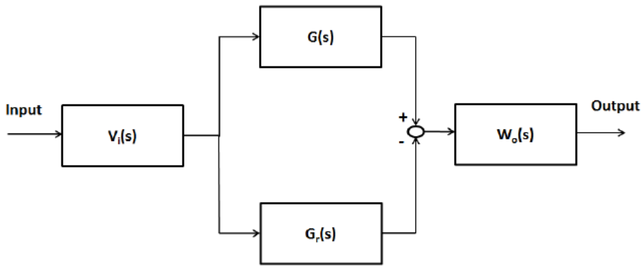

Figure 1.2: Input/Output FWMOR error system

weights. Problem is to estimate original system in a specific frequency interval in most cases. Using FWMOR for these cases, selecting weightings is also itself a issue[38] because, the engineer has to design weights for this frequency interval. Gawronski and Juang (GJ) [21] suggested a method, in which the frequency weighting have not explicity predefined that is known as frequency limited model order reduction, whereas approximation without input/output weighting is considered in a specific frequency range $[\omega_1, \omega_2]$. In that approach, Gramians are acquired for desirable frequency band. While, it can give unstable ROMs for stable system (same as Enns method [3]). Additionally, it provides no error bounds. Gugercin and Antoulas [GA] [38], Ghafoor and Sreeram [GS] [26], Imran and Ghafoor [IG] [47] and Imran et al. [48] have modified GJ [21] method to provide stable ROM and frequency response error bounds with objective to fulfilment the specific rank condition. Wang and Zilouchian (WZ) [40] developed a frequency limited model order reduction (FLMOR) method for discrete time system for retaining good approximation error in desirable frequency band. This technique does not ensure stability and not carry error bound of ROM for stable actual systems [26]. Furthermore, to resolve the limitation of WZ technique, the Ghafoor and Sreeram Algorithm 1 (GSA1) [26], Ghafoor and Sreeram Algorithm 2 (GSA2) [26] Imran and Ghafoor (IG) [49] and Hamid et al. [50] methods were proposed.

1.4 Problem Statment

Existing FLMOR method may provide unstable ROM, has no or weak error bound and provide large approximation error.

1.4.1 Summary of contribution

Many FLMOR methods [27] - [32] are proposed for both discrete and continuous time system that is represented either in state space or transfer function representation that provide always stable ROM, easily estimate error bound and yield commonly low approximation error.

1.5 Thesis Outline

Thesis consists of five different chapters. Each chapter's description is outlined here. In chapter 2, existing FLMOR techniques for continuous and discrete time system are described. These existing stability preserving methods overcome the instability problem of GJ [21] and WZ [40], IG (Continuous and Discrete) [47] [49], Imran et al.(Continuous) [48] and Hamid et al.(Discrete)[50]. In chapter 3 three novel methods are proposed for continuous and discrete time system which give stable ROM and yield better approximation error. In chapter 4 many numerical examples are discussed which show the effectiveness of the proposed methods in reducing approximation error for certain frequency range. Chapter 5 contain conclusion and some suggestion for future research. s

Chapter 2

Preliminaries

2.1 Continuous Time System Case

Consider linear continuous time invariant system represented in state space and in transfer function as

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t) \qquad (2.1) \\
G(s) &= C(sI - A)^{-1}B + D \qquad (2.2)
\end{aligned}
$$

where n^{th} minimal order realization $\{\ A \in R^{n\times n},\ B \in R^{n\times m}, C \in R^{p\times r}, D \in R^{p\times m}\}$ is with input/output m and p respectively. The problem of MOR is to find ROM

$$
\begin{aligned}
\dot{x}_{rom}(t) &= A_r x_r(t) + B_r u(t) \\
y_{rom}(t) &= C_r x_r(t) + D_r u(t) \qquad (2.3) \\
G_{rom}(s) &= C_r(sI - A_r)^{-1}B_r + D \qquad (2.4)
\end{aligned}
$$

that approximates the original system in a desired frequency band $[\Omega_1, \Omega_2]$ where $[\Omega_2 > \Omega_1]$, $\{ A_{rom} \in R^{r\times r}$, $B_{rom} \in R^{r\times m}$,$C_{rom} \in R^{p\times r}$,$D_{rom} \in R^{p\times m}\}$ and $r < n$. Consider the controllability Gramian P_c and the observability Gramian Q_o are obtained by using Parseval's Relationship

$$P_c = \frac{1}{2\pi}\int_{-\infty}^{\infty}(j\Omega I - A)^{-1}BB^T(-j\Omega I - A^T)^{-1}d\Omega \tag{2.5}$$

$$Q_o = \frac{1}{2\pi}\int_{-\infty}^{\infty}(-j\Omega I - A^T)^{-1}C^TC(j\Omega I - A)^{-1}d\Omega \tag{2.6}$$

The two different Gramians such as controllability and observability satisfying the following Lyapunov Equations

$$AP_c + P_cA^T + BB^T = 0 \tag{2.7}$$

$$A^TQ_o + Q_oA + CC^T = 0 \tag{2.8}$$

2.1.1 GJ [21] TECHNIQUE

Let the Gramians like observability Q_g and controllability P_g are defined for GJ [21] technique as $P_g = P(\Omega_2) - P(\Omega_1)$ and $Q_g = Q(\Omega_2) - Q(\Omega_1)$ respectively, for limited frequency interval. Gramians like observability and controllability satisfying the following Lyaunov equations

$$AP_g + P_gA^T + X_g = 0 \tag{2.9}$$

$$A^TQ_g + Q_gA + Y_g = 0 \tag{2.10}$$

$$P_g = \frac{1}{2\pi}\int_{-\Omega}^{\Omega}(j\Omega I - A)^{-1}BB^T(-j\Omega I - A^T)^{-1}d\Omega$$

$$Q_g = \frac{1}{2\pi}\int_{-\Omega}^{\Omega}(-j\Omega I - A^T)^{-1}C^TC(j\Omega I - A)^{-1}d\Omega$$

where

$$X_g = (S(\Omega_2) - S(\Omega_1))\, BB^T + BB^T\, (S^*(\Omega_2) - S^*(\Omega_1)) \quad (2.11)$$

$$Y_g = (S^*(\Omega_2) - S^*(\Omega_1))\, C^T C + C^T C\, (S(\Omega_2) - S(\Omega_1)) \quad (2.12)$$

$$S(\Omega) = \frac{j}{2\pi} \ln\left((j\Omega I + A)\, (-j\Omega I + A)^{-1} \right)$$

where $S^*(\Omega)$ is the $S(\Omega)$ conjugate transpose. Consider

$$T_g^T Q_g T_g = T_g^{-1} P_g T_g^{-T} = diag(\Sigma_1, \Sigma_2, ...\Sigma_n) \quad (2.13)$$

where $\Sigma_k \geq \Sigma_{k+1}$, $k = 1, 2, ..., n-1, \Sigma_r > \Sigma_{r+1}$. Contragredient matrix T_G is a that transform original system into balance realizations. The ROM is

$$G_{rom} = C_{rom}(sI - A_{rom})^{-1} B_{rom} + D_{rom}$$

obtained by partitioning the transformed realization.

$$T_g^{-1} A T_g = \begin{bmatrix} A_{11g} & A_{12g} \\ A_{21g} & A_{22g} \end{bmatrix}, \quad T_g^{-1} B = \begin{bmatrix} B_{1g} \\ B_{2g} \end{bmatrix}$$
$$CT_g = \begin{bmatrix} C_{1g} & C_{2g} \end{bmatrix}, \quad D \quad (2.14)$$

The symmetric matrices X_g and Y_g are defined as

$$X_g = USU^T = [U_1 \quad U_2] \begin{bmatrix} S_{g_1} & 0 \\ 0 & S_{g_2} \end{bmatrix} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} \quad (2.15)$$

$$Y_g = VRV^T = [V_1 \quad V_2] \begin{bmatrix} R_{g_1} & 0 \\ 0 & R_{g_2} \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \quad (2.16)$$

where

$$S_{g_1} = diag(s_1, ..., s_p) \geq 0, S_{g_2} = diag(s_{p+1}, ..., s_n) < 0,$$
$$R_{g_1} = diag(r_1, ..., r_q) \geq 0, R_{g_2} = diag(r_{q+1}, ..., r_n) < 0.$$

$p \leq n$ and $q \leq n$ are the index of positive eigenvalues of X_g and Y_g matrices respectively.

Remark1: For a desired frequency interval, the ROM achieved using GJ [21] technique is unstable. Symmetric matrices X_g and Y_g are not positive/semi positive definite, hence stability of ROM is not ensured [47].

2.1.2 Existing Stability Limited Frequency Techniques

Gugercin and Antoulas (GA) [38], Ghafoor and Seeram (GS) [26], Imran and Ghafoor (IG) [47] and Imran et al. [48] modified GJ [21] technique by making indefinite matrices positive/semi positive definiteness to ensure stability of ROM and error approximation in ROM. The modification done by IG [47] and Imran et al. [48] define fictitious input an output related matrices $B_Y \in \{B_a, B_s, B_i, B_f\}$ and $C_Y \in \{C_a, C_s, C_i, C_f\}$ respectively which satisfy following Lypunov Equations

$$AP_Y + P_Y A^T + B_Y B_Y^T = 0 \tag{2.17}$$
$$A^T Q_Y + Q_Y A + C_Y^T C_Y = 0 \tag{2.18}$$

The matrices B_Y and C_Y are obtained as:

$$B_a = U \begin{bmatrix} S_{g_1}^{1/2} & 0 \\ 0 & |S_{g_2}|^{1/2} \end{bmatrix}, C_a = \begin{bmatrix} R_{g_1}^{1/2} & 0 \\ 0 & |R_{g_2}|^{1/2} \end{bmatrix} V^T$$

$$B_s = U \begin{bmatrix} S_{g_1}^{1/2} & 0 \\ 0 & 0 \end{bmatrix}, C_s = \begin{bmatrix} R_{g_1}^{1/2} & 0 \\ 0 & 0 \end{bmatrix} V^T$$

$$B_i = U(S_g - s_n I)^{1/2}, \qquad C_i = (R_g - r_n I)^{1/2} V^T$$

Imran et al. [48] input and output matrices $B_{f_i} \in \{B_1, B_2, B_3\}$ and $C_{f_i} \in \{C_1, C_2, C_3\}$ are defined as

$$B_{f_i} = U \begin{bmatrix} S_{g_1}^{1/2} & 0 \\ 0 & S_{f_{2i}}^{1/2} \end{bmatrix}, \quad C_{f_i} = \begin{bmatrix} R_{g_1}^{1/2} & 0 \\ 0 & R_{f_{2i}}^{1/2} \end{bmatrix} V^T$$

$$S_{f_{2i}} = \begin{bmatrix} s_{l_i} & 0 & .... & 0 \\ 0 & S_{l_{i+1}} & ...... & 0 \\ 0 & 0 & .... & S_{n_i} \end{bmatrix}, \quad R_{f_{2i}} = \begin{bmatrix} r_{p_i} & 0 & .... & 0 \\ 0 & r_{p_{i+1}} & ...... & 0 \\ 0 & 0 & .... & r_{n_i} \end{bmatrix}$$

$$\begin{aligned} s_{l_1+q} &= s_{l+q+1}, s_{n_1} = s_n.s_{l+q}, \\ s_{l_2+q} &= -\frac{1}{2}(s_{l+q} + s_{l+q+1}, s_{n_2} = -\frac{1}{2}(s_n + s_{l+q}), \\ s_{l_3+q} &= (\frac{s_{l+q}}{s_{l+q+1}}), s_{n_3} = (\frac{s_n}{s_{l+q}})^{1/2}. \\ r_{p_1+h} &= r_{p+h}.r_{p+h+1}, r_{n_1} = r_n.r_{p+h}, \\ r_{p_2+h} &= -\frac{1}{2}(s_{p+h} + s_{p+h+1}, r_{n_2} = -\frac{1}{2}(r_n + r_{p+h}), \\ r_{p_3+h} &= (\frac{r_{p+h}}{s_{p+h+1}}), r_{n_3} = (\frac{r_n}{r_{p+h}})^{1/2} \end{aligned}$$

for $q = 0, 1, ..., n - l_i$ and for $h = 0, 1, ..., n - p_i$.

Remark 2: Since $X_g \leq B_y B_y^T$ equivalently $\lambda_i(B_y B_y^T) \geq \lambda_i(X_g)$ for every i, $\lambda_i(.)$ is the largest eigenvalues i^{th}, $\leq$ and $\geq$ represents less and greater than or equal to, $Y_g \leq C_y^T C_y$, $\{A, B_y, C_y\}$ is stable and minimal. These techniques yield frequency repones error bounds.

2.2 Discrete Time System Case

Consider a linear discrete time invariant system with following transfer function representation:

$$H(z) = C(zI - A)^{-1}B + D \tag{2.19}$$

where n^{th} minimal order realization is $\{A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}\}$ with input/output is m and p respectively. MOR problem is to compute ROM

$$H_{rom}(z) = C_{rom}(zI - A_{rom})^{-1}B_{rom} + D_{rom} \tag{2.20}$$

that approximates the original system in a desired frequency range $[\Omega_1, \omega_2]$ where $[\Omega_2 > \Omega_1]$, $\{ A_{rom} \in R^{r\times r}$, $B_{rom} \in R^{r\times m}$, $C_{rom} \in R^{p\times r}$, $D_{rom} \in R^{p\times m}\}$ and $r < n$.

Consider the controllability Gramian P_c and the observability Gramian Q_o are obtained by using Parseval's Relationship

$$P_c = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{j\Omega}I - A)^{-1}BB^T(e^{-j\Omega}I - A^T)^{-1}d\Omega \tag{2.21}$$

$$Q_o = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{-j\Omega}I - A^T)^{-1}C^TC(e^{j\Omega}I - A)^{-1}d\Omega \tag{2.22}$$

These Gramians are solution of following Lyapunov equations:

$$AP_cA^T - P_c + BB^T = 0 \tag{2.23}$$

$$A^TQ_oA - Q_o + CC^T = 0 \tag{2.24}$$

2.2.1 WZ [40] Technique

WZ [40] discrete time counterpart of GJ [21] technique introduce frequnecy limited concepts for discrete time systems. Let controllability P_{wz} and observability Q_{wz} Gramians are defined for WZ [40] technique as $P_{wz} = P(\Omega_2) - P(\Omega_1)$ and $Q_{wz} = Q(\Omega_2) - Q(\Omega_1)$ respectively, for limited frequency interval. The two different Gramians such as controllability and observability satisfying the following Lyapunov Equations:

$$AP_{wz}A^T - P_{wz} + X_{wz} = 0 \tag{2.25}$$

$$A^TQ_{wz}A - Q_{wz} + Y_{wz} = 0 \tag{2.26}$$

$$\begin{aligned} P_{wz} &= \frac{1}{2\pi}\int_{\delta\Omega}(e^{j\Omega}I - A)^{-1}BB^T(e^{-j\Omega}I - A^T)^{-1}d\Omega \\ Q_{wz} &= \frac{1}{2\pi}\int_{\delta\Omega}(e^{-j\Omega}I - A^T)^{-1}C^TC(e^{j\Omega}I - A)^{-1}d\Omega \end{aligned}$$

where

$$X_{wz} = \left(BB^TF^H + FBB^T\right) \tag{2.27}$$

$$Y_{wz} = \left(C^TCF + F^HC^TC\right) \tag{2.28}$$

$$F = -\frac{\Omega_2 - \Omega_1}{4\pi}I + \frac{1}{2\pi}\int_{\delta\Omega}(e^{j\Omega}I - A^T)^{-1}d\Omega$$

where F^H is the F Hermitian, $\delta\Omega$ is the intergation range $[\Omega_1, \Omega_2]$. Consider

$$T_{wz}^TQ_{wz}T_{wz} = T_{wz}^{-1}P_{wz}T_{wz}^{-T} = diag(\Sigma_1, \Sigma_2, ...\Sigma_n) \tag{2.29}$$

where $\Sigma_k \geq \Sigma_{k+1}$, $k = 1, 2, ..., n-1, \Sigma_r > \Sigma_{r+1}$. T_{WZ} is a contragredient matrix that transform original system into balanced realization. The ROM is

$$H_{rom} = C_{rom}(zI - A_{rom})^{-1}B_{rom} + D_{rom}$$

obtained by partitioning the transformed realization.

$$T_{wz}^{-1}AT_{wz} = \begin{bmatrix} A_{11z} & A_{12z} \\ A_{21z} & A_{22z} \end{bmatrix}, \; T_{wz}^{-1}B = \begin{bmatrix} B_{1z} \\ B_{2z} \end{bmatrix}$$
$$CT_{wz} = \begin{bmatrix} C_{1z} & C_{2z} \end{bmatrix}, \quad D \tag{2.30}$$

The symmetric matrices X_{wz} and Y_{wz} are defined as

$$X_{wz} = USU^T = [U_1 \quad U_2] \begin{bmatrix} S_{wz_1} & 0 \\ 0 & S_{wz_2} \end{bmatrix} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} \tag{2.31}$$

$$Y_{wz} = VRV^T = [V_1 \quad V_2] \begin{bmatrix} R_{wz_1} & 0 \\ 0 & R_{wz_2} \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \tag{2.32}$$

where

$$S_{wz_1} = diag(s_1, s_2, ..., s_p) \geq 0, S_{wz_2} = diag(s_{p+1}, s_{p+2}, ..., s_n) < 0,$$
$$R_{wz_1} = diag(r_1, r_2, ..., r_q) \geq 0, R_{wz_2} = diag(r_{q+1}, r_{q+2}, ..., r_n) < 0.$$

$p \leq n$ and $q \leq n$ are the index of positive eigenvalues of X_{wz} and Y_{wz} matrices respectively.

Remark3: For a desired frequency interval, the ROM achieved using WZ [40] technique is unstable. Symmetric matrices X_{wz} and Y_{wz} are not semipositive/positive definite, hence ROM stability is not ensured [49].

2.2.2 Existing Frequency Limited Stability Preserving Techniques

GSA1 [26], GSA2 [26], IG [49] and Hamid et al. [50] modified WZ [40] technique by making symmetric indefinite matrices X_{wz} and Y_{wz} positive/semipositive definite to ensure stability of ROM and error approximation in ROM. The modification done by IG [49] and Hamid et al. [50] define fictitious input an output related matrices $B_Y \in \{B_i, B_h\}$ and $C_Y \in \{C_i, C_h\}$ respectively, satisfying following Lyapunov Equations:

$$AP_YA^T - P_Y + B_YB_Y^T = 0 \tag{2.33}$$

$$A^TQ_YA - Q_Y + C_Y^TC_Y = 0 \tag{2.34}$$

The matrices B_Y and C_Y are obtained as:

$$B_{a1} = U \begin{bmatrix} S_{g_1}^{1/2} & 0 \\ 0 & |S_{g_2}|^{1/2} \end{bmatrix}, C_{a1} = \begin{bmatrix} R_{g_1}^{1/2} & 0 \\ 0 & |R_{g_2}|^{1/2} \end{bmatrix} V^T$$

$$B_{a2} = U \begin{bmatrix} S_{g_1}^{1/2} & 0 \\ 0 & 0 \end{bmatrix}, C_{a2} = \begin{bmatrix} R_{g_1}^{1/2} & 0 \\ 0 & 0 \end{bmatrix} V^T$$

$$B_i = U(S_{wz} - s_nI)^{1/2}, \qquad C_i = (R_{wz} - r_nI)^{1/2}V^T$$

Hamid et al. [50] input and output matrices $B_{h_i} \in \{B_1, B_2, B_3\}$ and $C_{h_i} \in \{C_1, C_2, C_3\}$ are defined as

Method I: i = 1

$$B_{h_1} = \begin{cases} U_{h_1}S_{h_1}^{1/2} \\ U_{h_2}(cos(S_{h_2}) - S_{h_2})^{1/2} \end{cases} \tag{2.35}$$

$$C_{h_1} = \begin{cases} R_{h_1}^{1/2} V_{h_1}^T \\ (cos(R_{h_2}) - R_{h_2})^{1/2} V_{H_l}^T \end{cases} \tag{2.36}$$

Method 2: i = 2

$$B_{h_2} = \begin{cases} U_{h_1} S_{h_1}^{1/2} \\ U_{h_2}((S_{h_2}/N \times \bar{r}) - S_{h_2})^{1/2} \end{cases} \tag{2.37}$$

$$C_{h_2} = \begin{cases} R_{h_1}^{1/2} V_{h_1}^T \\ ((R_{h_2}/N \times \bar{s}) - R_{h_2})^{1/2} V_{h_2}^T \end{cases} \tag{2.38}$$

Method 3: i = 3

$$B_{h_3} = \begin{cases} U_{h_1} S_{h_1}^{1/2} \\ U_{h_2}((S_{h_2}/\bar{r})^{1/N} - S_{h_2})^{1/2} \end{cases} \tag{2.39}$$

$$C_{h_3} = \begin{cases} R_{h_3}^{1/2} V_{h_3}^T \\ ((R_{h_2}/\bar{s})^{1/N} - R_{h_2})^{1/2} V_{h_2}^T \end{cases} \tag{2.40}$$

Although GJ [21] creates lowest frequency response errors in comparison with IG [47], and Imran et al. [48] techniques for continuous time systems, but it returns unstable ROM for stable original system. Moreover, all prior techniques ensure stability but at the cost of large approximation error. Similarly, in discrete time systems, WZ[40] yields lowest frequency response errors in comparison with IG [49], and Hamid et al. [50] techniques, but it returns unstable ROM for stable original system. Likewise, all prior techniques ensure stability but at the cost of large approximation error. In order to minimize the approximation error, some new methods are proposed which will not only guarantee stability, give error bound but also mostly provide low frequency response approximation error when compared to prior stability preserving methods.

Chapter 3

Main Result

3.1 Proposed Techniques for Continuous Time Systems Case

The precedent frequency limited techniques ensure stability by making symmetric matrices X_g (2.11) and Y_g (2.12) positive/semipositive definite by using various methods. Although GJ [21] yields lowest frequency response errors in comparison with IG [47], and Imran et al. [48] methods, but it returns unstable ROM for stable original system. Moreover, all prior methods persevere stability but at the cost of large approximation error. In order to minimize the approximation error, some new methods are proposed that will guarantee stability, give error bounds as well as mostly have low frequency response approximation error when these proposed methods compared to prior stability preserving methods.

3.1.1 Proposed Methods

Consider the two different Gramian such as controllability P_{H_j} and observability Q_{H_j} satisfying the following Lyapunov equations:

$$AP_{H_j} + P_{H_j}A^T + B_{H_j}B_{H_j}^T = 0 \tag{3.1}$$

$$A^T Q_{H_j} + Q_{H_j}A + C_{H_j}^T C_{H_j} = 0 \tag{3.2}$$

where $j = 1, 2, 3$. The different input/output related matrices are specify for indefinite symmetric X_g and Y_g matrices as B_{H_j} and C_{H_j} as follows:

Method 1: for j = 1

$$B_{H_1} = \begin{cases} U_{H_1} S_{H_1}^{1/2} \\ U_{H_2}(exp(S_{H_2}/trace(S)))^{1/2} \end{cases} \tag{3.3}$$

$$C_{H_1} = \begin{cases} R_{H_1}^{1/2} V_{H_1}^T \\ (exp(R_{H_2}/trace(R)))^{1/2} V_{H_2}^T \end{cases} \tag{3.4}$$

Method 2: for j = 2

$$B_{H_2} = \begin{cases} U_{H_1} S_{H_1}^{1/2} \\ U_{H_2}((S_{H_2} \times S_{H_2})/Sn))^{1/2} \end{cases} \tag{3.5}$$

Sn=abs(product of all negative eigenvalues)

$$C_{H_2} = \begin{cases} R_{H_1}^{1/2} V_{H_1}^T \\ ((R_{H_2} \times R_{H_2})/Rn)^{1/2} V_{H_2}^T \end{cases} \tag{3.6}$$

Rn=abs(product of all negative eigenvalues)

Method 3: for j = 3

$$B_{H_3} = \begin{cases} U_{H_1} S_{H_1}^{1/2} \\ U_{H_2}(sec(S_{H_2}) - (sn))^{1/2} \end{cases} \tag{3.7}$$

$$C_{H_3} = \begin{cases} R_{H_1}^{1/2} V_{H_1}^T \\ (sec(R_{H_2}) - (rn))^{1/2} V_{H_2}^T \end{cases} \tag{3.8}$$

where $Sn = |s_{p+1} \times s_{p+2} \times ...... \times sn|$, $Rn = |r_{q+1} \times r_{q+2} \times ...... \times r_n|$. The terms U_{H_1}, U_{H_2}, S_{H_1}, S_{H_2}, V_{H_1}, V_{H_2}, R_{H_1}, R_{H_2} are obtained from following symmetric matrices,

$$X_g = [USU^T] = [U_{H_1} \quad U_{H_2}] \begin{bmatrix} S_{H_1} & 0 \\ 0 & S_{H_2} \end{bmatrix} \begin{bmatrix} U_{H_1}^T \\ U_{H_2}^T \end{bmatrix} \tag{3.9}$$

$$Y_g = [VRV^T] = [V_{H_1} \quad V_{H_2}] \begin{bmatrix} R_{H_1} & 0 \\ 0 & R_{H_2} \end{bmatrix} \begin{bmatrix} V_{H_1}^T \\ V_{H_2}^T \end{bmatrix} \tag{3.10}$$

where

$$S_{H_1} = diag(s_1, ..., s_p), S_{H_2} = diag(s_{p+1}, s_{p+2}....., s_n),$$
$$R_{H_1} = diag(r_1, ..., r_q), R_{H_2} = diag(r_{q+1}, r_{q+2}....., r_n).$$

Remark 5: When $X_g > 0$ and $Y_g > 0$, $B_{H_j} = S_{H_j} S_{H_j}^{1/2}$ and $C_{H_j} = R_{H_j}^{1/2} V_{H_j}^T$. Let T_{H_j} (transformation matrix) be obtained via simultaneously diagonalizing the Gramians

$$T_{H_j}^T Q_{H_j} T_{H_j} = T_{H_j}^{-1} P_{H_j} T_{H_j}^{-T} = diag(\Sigma_1, \Sigma_2 \cdots \Sigma_n) \tag{3.11}$$

where $\Sigma_k \geq \Sigma_{k+1}$, $k = 1, 2, 3, ..., n-1, \Sigma_r > \Sigma_{r+1}$. A ROM $\{A_{11}, B_1, C_1, D\}$ is acquired by transforming and portioning the realization as

$$T_{H_j}^{-1} A T_{H_j} = \begin{bmatrix} A_{11H} & A_{12H} \\ A_{21H} & A_{22H} \end{bmatrix} \; T_{H_j}^{-1} B = \begin{bmatrix} B_{1H} \\ B_{2H} \end{bmatrix}$$
$$CT_{H_j} = \begin{bmatrix} C_1 H & C_2 H \end{bmatrix}, \quad D \tag{3.12}$$

Remark 6: Each input matrix $B_{H_1}, B_{H_2}, B_{H_3}$ and output matrix $C_{H_1}, C_{H_2}, C_{H_3}$ ensure semipositive/ positive definiteness of the input/output related matrices, consequently semipositive/positive definiteness of $P_{H_1}, P_{H_2}, P_{H_3}$ and $Q_{H_1}, Q_{H_2}, Q_{H_3}$ in a different way. This guides towards a new transformation matrices $T_{H_1}, T_{H_2}, T_{H_3}$ which consequently result in three distinct stability preserving MOR methods.

Remark 7: Since $X_g \leq B_{H_j}B_{H_j}^T$, $Y_g \leq C_{H_j}^T C_{H_j}$, $P_{H_j} > 0$ and $Q_{H_j} > 0$. Hence, (A, B_{H_j}, C_{H_j}) realization is minimal and also ensured the ROM stability.

3.1.2 Error Bounds

Theorem 1: For the proposed methods, the error bound are given hold if the rank conditions $rank\begin{bmatrix} B_{H_j} & B \end{bmatrix} = rank\begin{bmatrix} B_{H_j} \end{bmatrix}$ and $rank\begin{bmatrix} C_{H_j} \\ C \end{bmatrix} = rank\begin{bmatrix} C_{H_j} \end{bmatrix}$ is satisfying

$$\|G(s) - G_r(s)\|_\infty \leq 2\|L_{H_j}\|_\infty \|K_{H_j}\|_\infty \sum_{i=r+1}^{n} \sigma_i$$

for j=1

$$L_{H_1} = \begin{cases} CV_{H_1} R_{H_1}^{-1/2} \\ CV_{H_2}(exp(R_{H_2}/trace(R)))^{-1/2} \end{cases} \tag{3.13}$$

$$K_{H_1} = \begin{cases} S_{H_1}^{-1/2} U_{H_1}^T B \\ (exp(S_{H_2}/trace(S)))^{-1/2} U_{H_2}^T B \end{cases} \tag{3.14}$$

for j=2

$$L_{H_2} = \begin{cases} CV_{H_1} R_{H_1}^{-1/2} \\ CV_{H_2}((R_{H_2} \times R_{H_2})/Rn)^{-1/2} \end{cases} \tag{3.15}$$

$$K_{H_2} = \begin{cases} S_{H_1}^{-1/2} U_{H_1}^T B \\ ((S_{H_2} \times S_{H_2})/Sn)^{-1/2} U_{H_2}^T B \end{cases} \tag{3.16}$$

for j=3

$$L_{H_3} = \begin{cases} CV_{H_1}R_{H_1}^{-1/2} \\ CV_{H_2}(sec(R_{H_2}) - (rn))^{-1/2} \end{cases} \quad (3.17)$$

$$K_{H_3} = \begin{cases} S_{H_1}^{-1/2}U_{H_1}^{T}B \\ (sec(S_{H_2}) - (sn))^{-1/2}U_{H_2}^{T}B \end{cases} \quad (3.18)$$

Proof: Since $rank\begin{bmatrix} B_{H_j} & B \end{bmatrix} = rank\begin{bmatrix} B_{H_j} \end{bmatrix}$ and $rank\begin{bmatrix} C_{H_j} \\ C \end{bmatrix} = rank\begin{bmatrix} C_{H_j} \end{bmatrix}$, the relationships $B = B_{H_j}K_{H_j}$ and $C = L_{H_j}C_{H_j}$ hold. By partitioning $B_H = \begin{bmatrix} B_{H_1} \\ B_{H_2} \end{bmatrix}$, $C_H = \begin{bmatrix} C_{H_1} & C_{H_2} \end{bmatrix}$ and substituting $B_1 = B_{H_1}K_{H_j}$, $C_1 = L_{H_j}C_{H_1}$ respectively yields

$$\begin{aligned} \|G(s) - G_r(s)\|_\infty &= \|C(sI - A)^{-1}B - C_1(sI - A_{11H})^{-1}B_1H\|_\infty \\ &= \|L_{H_j}C_{H_j}(sI - A)^{-1}B_{H_j}K_{H_j} - L_{H_j}C_{H_1}(sI - A_{11H})^{-1}B_{H_1}K_{H_j}\|_\infty \\ &= \|L_{H_j}(C_H(sI - A)^{-1}B_H - C_{H_1}(sI - A_{11H})^{-1}B_{H_1})K_{H_j}\|_\infty \\ &\leq \|L_{H_j}\|_\infty\|(C_H(sI - A)^{-1}B_H - C_{H_1}(sI - A_{11H})^{-1}B_{H_1})\|_\infty\|K_{H_j}\|_\infty \end{aligned}$$

If the ROM $\{A_{11H}, B_{H_1}, C_{H_1}\}$ is acquired by partitioning of $\{A, B_H, C_H\}$ balanced realization , we have from [3]

$$\|(C_H(sI - A)^{-1}B_H - C_{H_1}(sI - A_{11H})^{-1}B_{H_1})\|_\infty \leq 2\sum_{k=r+1}^{n}\sigma_i.$$

Thus,

$$\|G(s) - G_{rom}(s)\|_\infty \leq 2\|L_{H_j}\|_\infty\|K_{H_j}\|_\infty\sum_{i=r+1}^{n}\Sigma_i$$

Hence the results follows.

Remark 8: Three choices of $K_{H_1}, K_{H_2}, K_{H_3}$ and $L_{H_1}, L_{H_2}, L_{H_3}$ form are basis of error bound derivation for proposed methods.

Algorithm

1. Use equations (2.11) and (2.12) to compute the value of X_g and Y_g.

2. Use equations (3.3),(3.5),(3.7) and (3.4),(3.6),(3.8) to determine the values of $B_{H_1}, B_{H_2}, B_{H_3}$ and $C_{H_1}, C_{H_2}, C_{H_3}$ respectively.

3. The values of P_{H_j} and Q_{H_j} are computed using equation (3.1) and (3.2).

4. Use equation (3.11) to compute T_{H_j}.

5. Compute balanced realization to obtain ROM by using equation (3.12).

3.2 Proposed Methods for Discrete Time Systems Case

The precedent frequency limited techniques ensure stability by making symmetric matrices X_{wz} (2.27) and Y_{wz} (2.28) positive/semipositive definite by using various methods. Although WZ [40] yields lowest frequency response errors in comparison with IG [49], and Hamid et al. [50] techniques, but it returns unstable ROM for stable original system. Moreover, all prior techniques ensure stability but at the cost of large approximation error. In order to minimize the approximation error, some new methods are proposed that will guarantee stability, give error bounds as well as mostly have low frequency response approximation error when these proposed techniques compared to prior stability preserving methods.

3.2.1 Proposed Methods

Consider the different controllable Gramian P_{H_j} and observable Gramian Q_{H_j} satisfying following Lyapunov equations:

$$AP_{H_j}A^T - P_{H_j} + B_{H_j}B_{H_j}^T = 0 \qquad (3.19)$$

$$A^TQ_{H_j}A - Q_{H_j} + C_{H_j}^TC_{H_j} = 0 \qquad (3.20)$$

where $j = 1, 2, 3$. The different input/output related matrices are defined for indefinite symmetric X_{wz} and Y_{wz} matrices as B_{H_j} and C_{H_j} as follows:

Technique 1: for j = 1

$$B_{H_1} = \begin{cases} U_{H_1} S_{H_1}^{1/2} \\ U_{H_2}(exp(S_{H_2}/trace(S)))^{1/2} \end{cases} \tag{3.21}$$

$$C_{H_1} = \begin{cases} R_{H_1}^{1/2} V_{H_1}^{T} \\ (exp(R_{H_2}/trace(R)))^{1/2} V_{H_2}^{T} \end{cases} \tag{3.22}$$

Technique 2: for j = 2

$$B_{H_2} = \begin{cases} U_{H_1} S_{H_1}^{1/2} \\ U_{H_2}((S_{H_2} \times S_{H_2})/Sn))^{1/2} \end{cases} \tag{3.23}$$

Sn=abs(product of all negative eigenvalues)

$$C_{H_2} = \begin{cases} R_{H_1}^{1/2} V_{H_1}^{T} \\ ((R_{H_2} \times R_{H_2})/Rn)^{1/2} V_{H_2}^{T} \end{cases} \tag{3.24}$$

Rn=abs(product of all negative eigenvalues)

Technique 3: for j = 3

$$B_{H_3} = \begin{cases} U_{H_1} S_{H_1}^{1/2} \\ U_{H_2}(sec(S_{H_2}) - (sn))^{1/2} \end{cases} \tag{3.25}$$

$$C_{H_3} = \begin{cases} R_{H_1}^{1/2} V_{H_1}^{T} \\ (sec(R_{H_2}) - (rn))^{1/2} V_{H_2}^{T} \end{cases} \tag{3.26}$$

where$Sn = |s_{p+1} \times s_{p+2} \times ...... \times sn|$, $Rn = |r_{q+1} \times r_{q+2} \times ...... \times r_n|$. The terms U_{H_1}, U_{H_2}, S_{H_1}, S_{H_2}, V_{H_1}, V_{H_2}, R_{H_1}, R_{H_2} are obtained from following symmetric matrices,

$$X_{wz} = [USU^T] = [U_{H_1} \quad U_{H_2}] \begin{bmatrix} S_{H_1} & 0 \\ 0 & S_{H_2} \end{bmatrix} \begin{bmatrix} U_{H_1}^T \\ U_{H_2}^T \end{bmatrix} \tag{3.27}$$

$$Y_{wz} = [VRV^T] = [V_{H_1} \quad V_{H_2}] \begin{bmatrix} R_{H_1} & 0 \\ 0 & R_{H_2} \end{bmatrix} \begin{bmatrix} V_{H_1}^T \\ V_{H_2}^T \end{bmatrix} \tag{3.28}$$

where

$$S_{H_1} = diag(s_1, ..., s_p), S_{H_2} = diag(s_{p+1}, s_{p+2}....., s_n),$$
$$R_{H_1} = diag(r_1, ..., r_q), R_{H_2} = diag(r_{q+1}, r_{q+2}....., r_n).$$

Remark 9: When $X_{wz} > 0$ and $Y_{wz} > 0$, $B_{H_j} = S_{H_j} S_{H_j}^{1/2}$ and $C_{H_j} = R_{H_j}^{1/2} V_{H_j}^T$. Let T_{H_j} (transformation matrix) be obtained via simultaneously diagonalizing the Gramians

$$T_{H_j}^T Q_{H_j} T_{H_j} = T_{H_j}^{-1} P_{H_j} T_{H_j}^{-T} = diag(\Sigma_1, \Sigma_2 \cdots \Sigma_n) \tag{3.29}$$

where $\Sigma_k \geq \Sigma_{k+1}, k = 1, 2, 3, ..., n-1, \Sigma_r > \Sigma_{r+1}$. A ROM $\{A_{11H}, B_{1H}, C_1, D\}$ is acquired by portioning the transformed realization as

$$T_{H_j}^{-1} A T_{H_j} = \begin{bmatrix} A_{11H} & A_{12H} \\ A_{21H} & A_{22H} \end{bmatrix} T_{H_j}^{-1} B = \begin{bmatrix} B_{1H} \\ B_{2H} \end{bmatrix}$$
$$CT_{H_j} = \begin{bmatrix} C_1 H & C_2 H \end{bmatrix}, \quad D \tag{3.30}$$

Remark 10: Each input matrix $B_{H_1}, B_{H_2}, B_{H_3}$ and output matrix $C_{H_1}, C_{H_2}, C_{H_3}$ ensure semipositive/positive definiteness of input/output related matrices, consequently semipositive/positive definiteness of $P_{H_1}, P_{H_2}, P_{H_3}$ and $Q_{H_1}, Q_{H_2}, Q_{H_3}$ in a different way. That guides toward a new transformation matrices $T_{H_1}, T_{H_2}, T_{H_3}$ which consequently result in three new stability preserving MOR methods.

Remark 11: Since $X_{wz} \leq B_{H_j} B_{H_j}^T$, $Y_{wz} \leq C_{H_j}^T C_{H_j}$, $P_{H_j} > 0$ and $Q_{H_j} > 0$. Hence, the realization is minimal (A, B_{H_j}, C_{H_j}) and ensure ROM stability.

3.2.2 Error Bounds

Theorem 1: For the proposed methods, the given error bound hold if the rank conditions $rank \begin{bmatrix} B_{H_j} & B \end{bmatrix} = rank \begin{bmatrix} B_{H_j} \end{bmatrix}$ and $rank \begin{bmatrix} C_{H_j} \\ C \end{bmatrix} = rank \begin{bmatrix} C_{H_j} \end{bmatrix}$ is satisfying

$$\|H(z) - H_{rom}(z)\|_\infty \leq 2\|L_{H_j}\|_\infty \|K_{H_j}\|_\infty \sum_{i=r+1}^{n} \sigma_i$$

for j=1

$$L_{H_1} = \begin{cases} CV_{H_1} R_{H_1}^{-1/2} \\ CV_{H_2}(exp(R_{H_2}/trace(R)))^{-1/2} \end{cases} \tag{3.31}$$

$$K_{H_1} = \begin{cases} S_{H_1}^{-1/2} U_{H_1}^T B \\ (exp(S_{H_2}/trace(S)))^{-1/2} U_{H_2}^T B \end{cases} \tag{3.32}$$

for j=2

$$L_{H_2} = \begin{cases} CV_{H_1} R_{H_1}^{-1/2} \\ CV_{H_2}((R_{H_2} \times R_{H_2})/Rn)^{-1/2} \end{cases} \tag{3.33}$$

$$K_{H_2} = \begin{cases} S_{H_1}^{-1/2} U_{H_1}^T B \\ ((S_{H_2} \times S_{H_2})/Sn)^{-1/2} U_{H_2}^T B \end{cases} \tag{3.34}$$

for j=3

$$L_{H_3} = \begin{cases} CV_{H_1} R_{H_1}^{-1/2} \\ CV_{H_2}(sec(R_{H_2}) - (rn))^{-1/2} \end{cases} \tag{3.35}$$

$$K_{H_3} = \begin{cases} S_{H_1}^{-1/2} U_{H_1}^T B \\ (sec(S_{H_2}) - (sn))^{-1/2} U_{H_2}^T B \end{cases} \tag{3.36}$$

Proof: Since $rank \begin{bmatrix} B_{H_j} & B \end{bmatrix} = rank \begin{bmatrix} B_{H_j} \end{bmatrix}$ and $rank \begin{bmatrix} C_{H_j} \\ C \end{bmatrix} = rank \begin{bmatrix} C_{H_j} \end{bmatrix}$, the relationships $B = B_{H_j} K_{H_j}$ and $C = L_{H_j} C_{H_j}$ hold. By partitioning $B_H = \begin{bmatrix} B_{H_1} \\ B_{H_2} \end{bmatrix}$, $C_H = \begin{bmatrix} C_{H_1} & C_{H_2} \end{bmatrix}$ and substituting $B_1 = B_{H_1} K_{H_j}$, $C_1 = L_{H_j} C_{H_1}$ respectively yields

$$\begin{aligned}
\|H(z) - H_r om(z)\|_\infty &= \|C(zI - A)^{-1}B - C_1(zI - A_{11H})^{-1}B_1\|_\infty \\
&= \|L_{H_j} C_{H_j}(zI - A)^{-1} B_{H_j} K_{H_j} - L_{H_j} C_{H_1}(zI - A_{11H})^{-1} B_{H_1} K_{H_j}\|_\infty \\
&= \|L_{H_j}(C_H(zI - A)^{-1} B_H - C_{H_1}(zI - A_{11H})^{-1} B_{H_1}) K_{H_j}\|_\infty \\
&\leq \|L_{H_j}\|_\infty \|(C_H(zI - A)^{-1} B_H - C_{H_1}(zI - A_{11H})^{-1} B_{H_1})\|_\infty \|K_{H_j}\|_\infty
\end{aligned}$$

If ROM is $\{A_{11H}, B_{H_1}, C_{H_1}\}$ acquired by partitioning of balanced realization $\{A, B_H, C_H\}$, we have from [3]

$$\|(C_H(zI - A)^{-1} B_H - C_{H_1}(zI - A_{11H})^{-1} B_{H_1})\|_\infty \leq 2 \sum_{k=r+1}^{n} \sigma_i.$$

Thus,

$$\|H(z) - H_r om(z)\|_\infty \leq 2\|L_{H_j}\|_\infty \|K_{H_j}\|_\infty \sum_{i=r+1}^{n} \Sigma_i$$

Hence the results follows.

Remark 12: Three choices of $K_{H_1}, K_{H_2}, K_{H_3}$ and $L_{H_1}, L_{H_2}, L_{H_3}$ form are basis of error bound derivation for proposed methods.

Algorithm

1. Use equations (2.27) and (2.28) to compute the value of X_{wz} and Y_{wz}.
2. Use equations (3.23),(3.25),(3.27) and (3.24),(3.26),(3.28) to determine the values of $B_{H_1}, B_{H_2}, B_{H_3}$ and $C_{H_1}, C_{H_2}, C_{H_3}$ respectively.

3. The values of P_{H_j} and Q_{H_j} are computed using equation (3.21) and (3.22).

4. Use equation (3.29) to compute T_{H_j}.

5. Compute balanced realization to obtain ROM by using equation (3.30).

Chapter 4

Numerical Simulation

4.1 Examples for Continuous Time Systems Case

This section contains numerical examples showing frequency response error of existing and proposed techniques in the desire frequency range.

Example 1: Let a stable 6^{th} order original system with given state-space representation

$$A = \begin{bmatrix} -20 & -155 & -586 & -1115 & -1034 & -390 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$C = \begin{bmatrix} 0 & 0 & -44.1 & -204 & -934 & 90 \end{bmatrix}$$

$$D = 0$$

in the desirable frequency range $[35 - 49]$ rad/s. Fig 4.1 compares the singular values of reduction error (frequency response error) $\sigma[G(s) - G_r(s)]$, where $G_r(S)$ is 3^{rd} order ROM achieved by using GJ [21], IG [47], Imran et al. [48] and proposed techniques in the desirable frequency range. Fig. 4.2 shows the closed view of Fig 4.1. It can be observed from Fig 4.2 that proposed techniques mostly give low approximation error when these are compared with prior stability preserving methods IG [47] and Imran et al [48].

Example 2: Let a stable 8^{th} order original system with given state-space representation

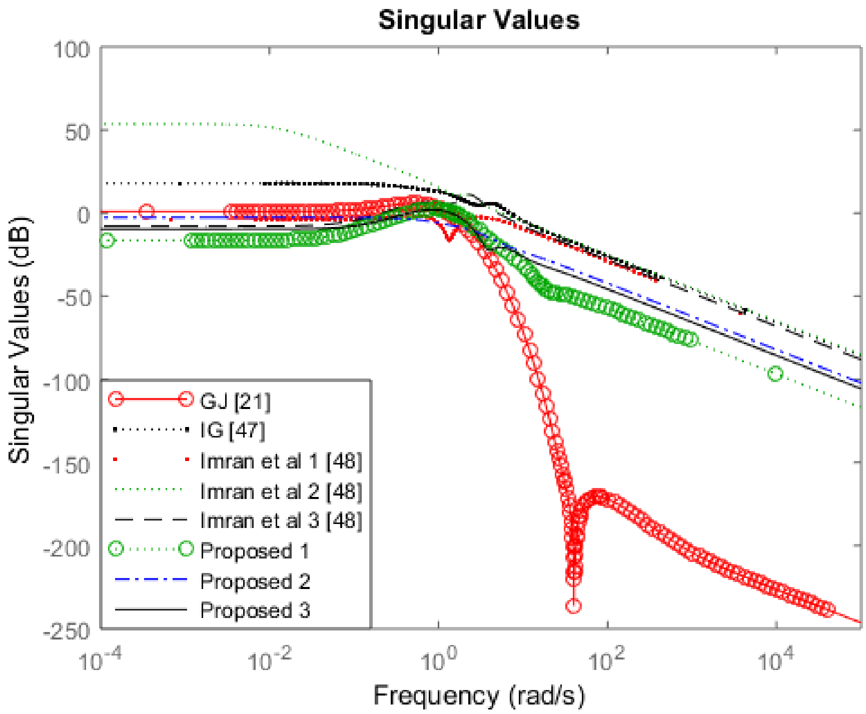

Figure 4.1: Frequency response error with comparison for example 1.

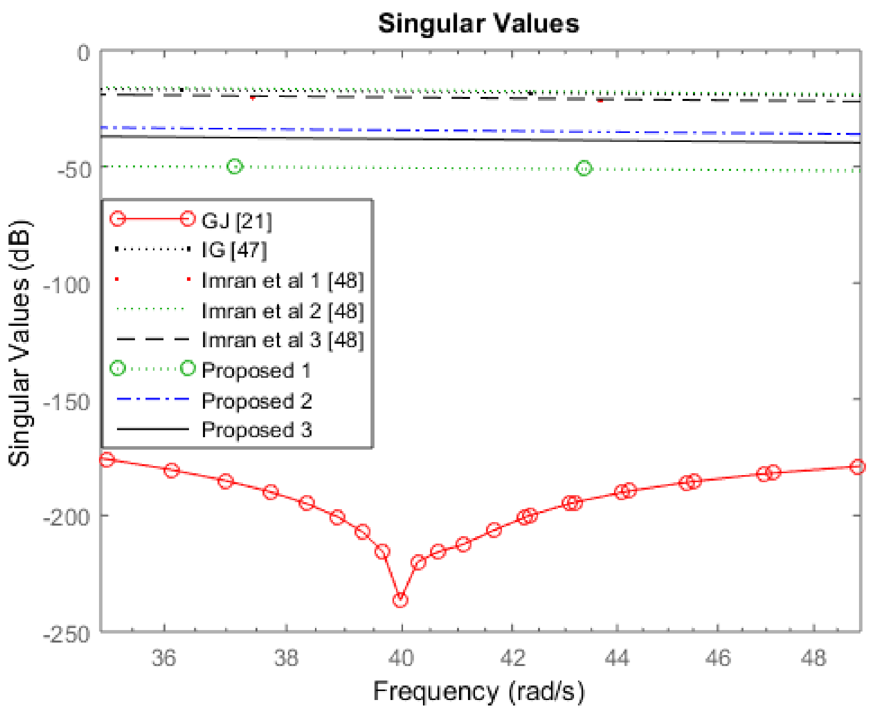

Figure 4.2: Closed view of error plot in desirable frequency interval for example 1.

$$A = 1e^3 \times \begin{bmatrix} -1.4 & -325 & -321.9 & -2783.0 & -1449.0 & -6581.0 & -1284.0 & -4101.0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 468.4 & -0.0019 & 0 & 0 & -0.0002 \end{bmatrix}$$

$$D = 0$$

in a desirable frequency interval $[12 - 17]$ rad/s. Fig 4.3 presents the comparison of singular values of reduction error (frequency response error) $\sigma[G(s) - G_r(s)]$, where $G_r(s)$ is 3^{rd} order ROM achieved by GJ [21], IG [47], Imran el at. [48] and propose techniques. Fig 4.4 shows the closed view of Fig 4.3 in the desirable frequency band. It can be observed from the Fig 4.4 that purposed techniques mostly have low approximation error when these are compared with prior stability preserving methods IG [47] and Imran et al [48] in the desirable frequency range. Table I presents the poles of 3^{rd} order ROM obtained by GJ [21], IG [47], Imran et al [48] and proposed techniques. It can be seen that GJ [21] yield unstable ROM with pole location at $s = 6.2981$.

Example 3: Suppose a chebyshev type 2 stable high pass filter of 16^{th} order have stop band ripple of 13 and stop band edge frequency of 28Hz with desired frequency interval [17 - 22] rad/s. Fig 4.5 present $\sigma[G(s) - G_r(s)]$, where $G_r(s)$ is of 4^{th} order ROM obtained using GJ [21], IG [47] and Imran et al. [48] and proposed methods in the desirable frequency range. Fig 4.6 represents closed view

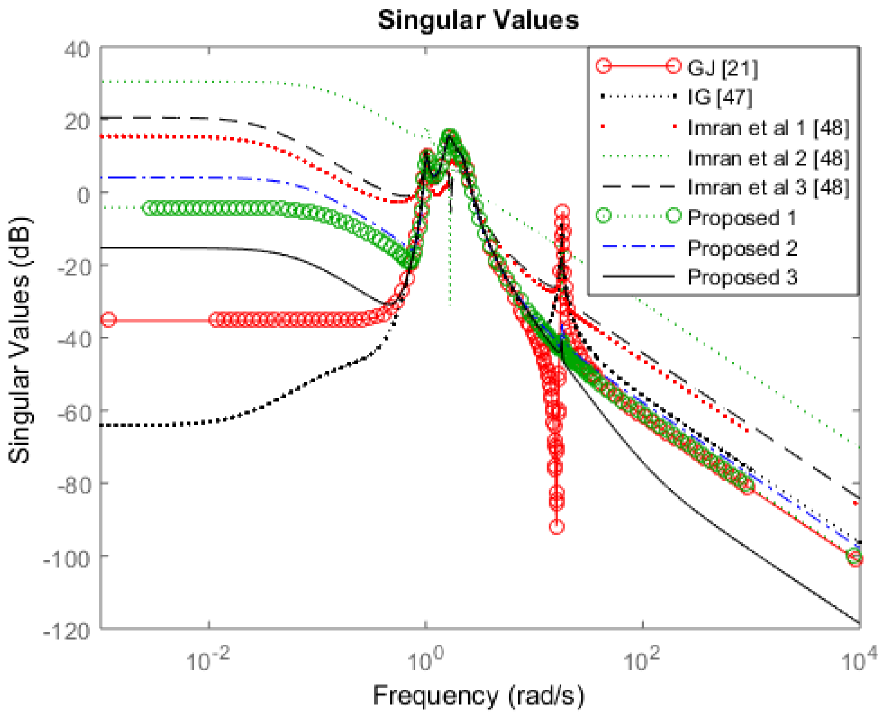

Figure 4.3: Frequency response error with comparison for example 2.

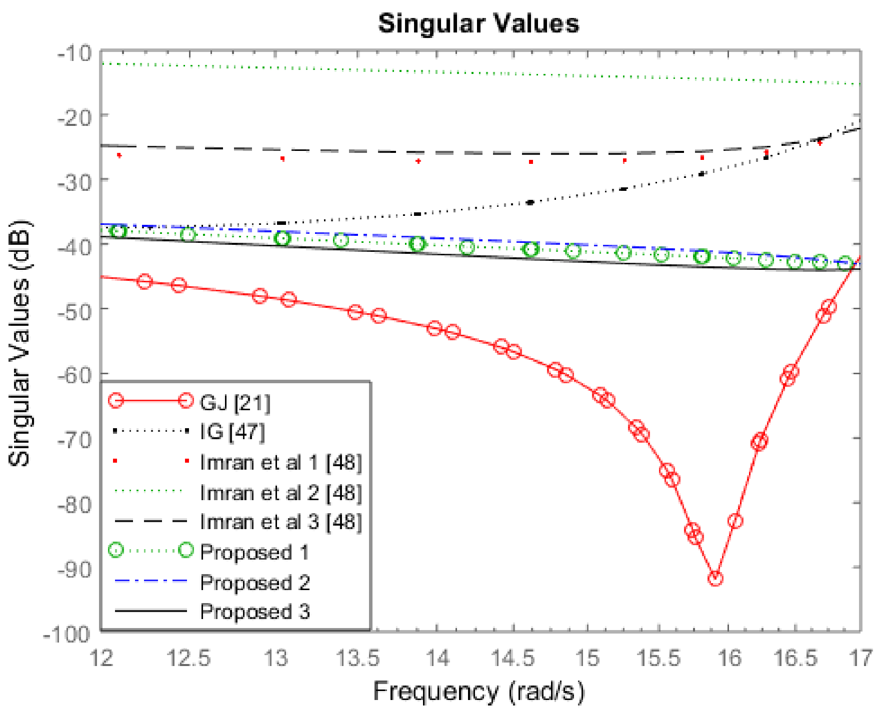

Figure 4.4: Closed view of error plot in desirable frequency interval for example 2.

Table 4.1: Roots of 3^{rd} order ROM of EXAMPLE 2

Techniques	**Roots**
GJ [21]	-0.1115 ± 18.3124i, 6.2981
IG [47]	-0.21.8 ± 17.7694i, -0.1823
Imran et al 1 [48]	-0.1161 ± 1.8204i, -0.1342
Imran et al 2 [48]	-0.0871 ± 1.7197i, -0.0339
Imran et al 3 [48]	-0.0899 ± 1.6345i, -0.00377
Proposed 1	-0.2094 ± 17.7704i, -0.1370
Proposed 2	-0.2087 ± 17.7706i, -0.0781
Proposed 3	-0.2095 ± 17.7704i, -0.1236

of Fig 4.5. It can be clear from the Fig 4.6 that ROM acquired from proposed techniques mostly have low approximation error as compared with other existing methods IG [47] and Imran et al. [48] in the desirable frequency band.

Example 4: Suppose the 30^{th} order, bandpass chebyshev type 1 stable filter which passes frequencies between 13 to 32Hz with 25 dB of ripple in the passband in the desired frequency interval [52 - 57.7] rad/s. Fig 4.7 present $\sigma[G(s)-G_r(s)]$, where $G_r(s)$ is of 4^{th} order ROM obtained using GJ [21], IG [47] and Imran et al. [48] and proposed methods. Fig 4.8 shows closed view of Fig 4.7 in the desired frequency band. It can be clear from the Fig 4.8 that ROM acquired from proposed methods mostly give low approximation error when these are compared to prior stability preserving methodss IG [47] and Imran et al. [48] in the desirable frequency band.

Example 5: Consider a high pass chebyshev type 2 stable filter of order 20^{th}

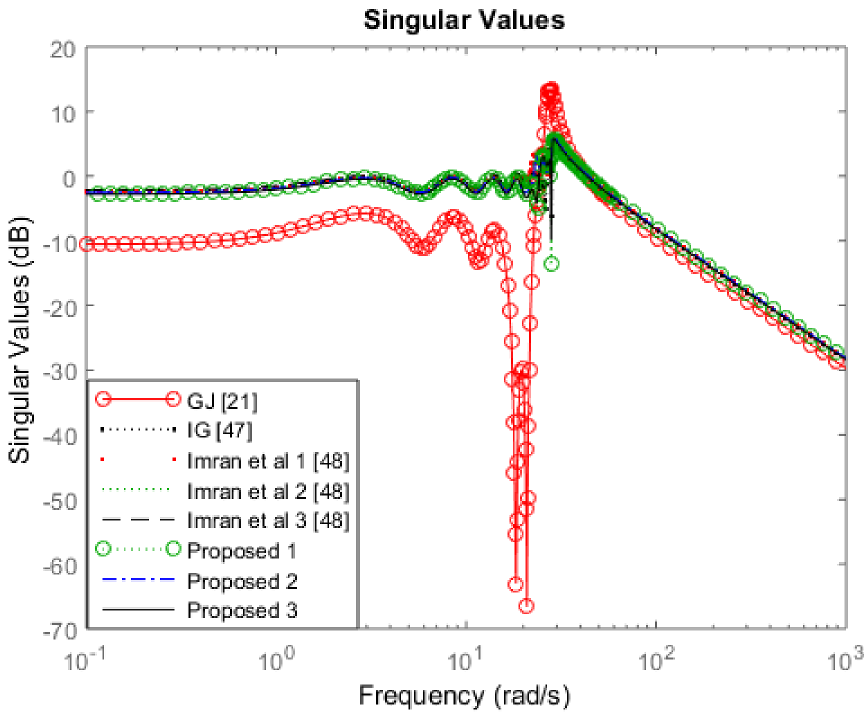

Figure 4.5: Frequency response error with comparison for example 3.

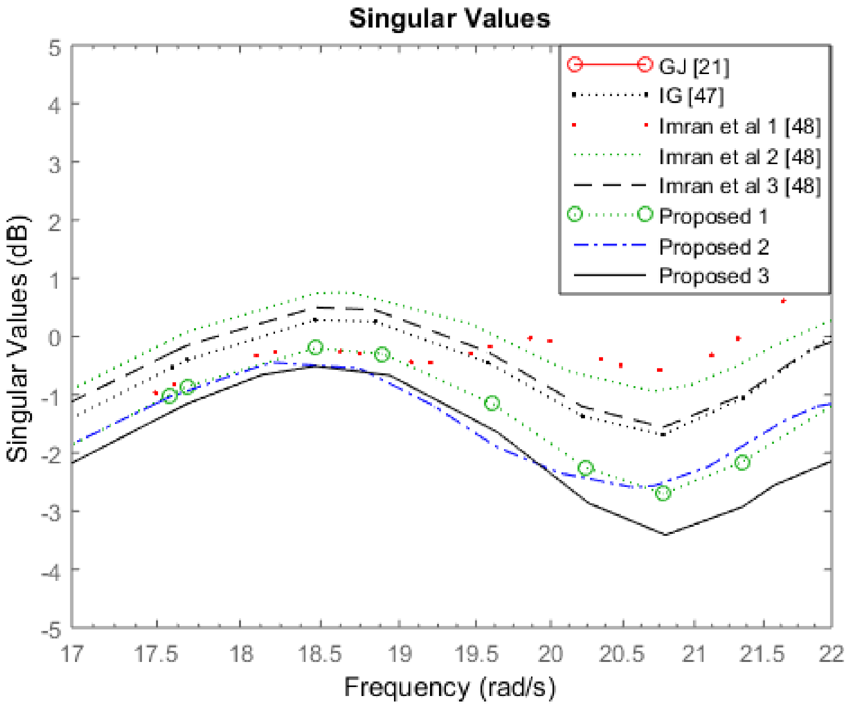

Figure 4.6: Closed view of error plot in desirable frequency interval for example 3.

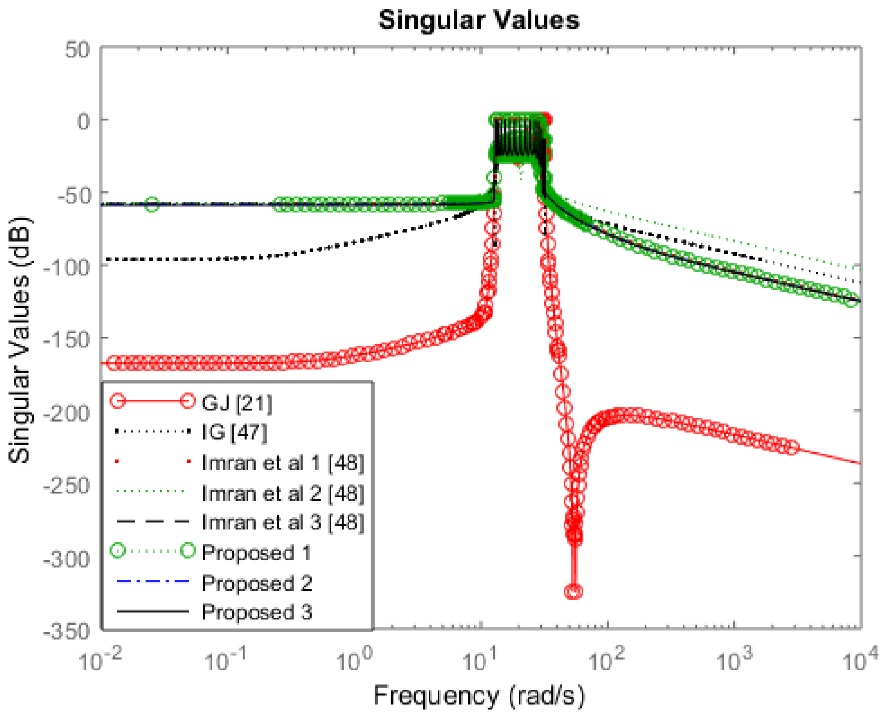

Figure 4.7: Frequency response error with comparison for example 4.

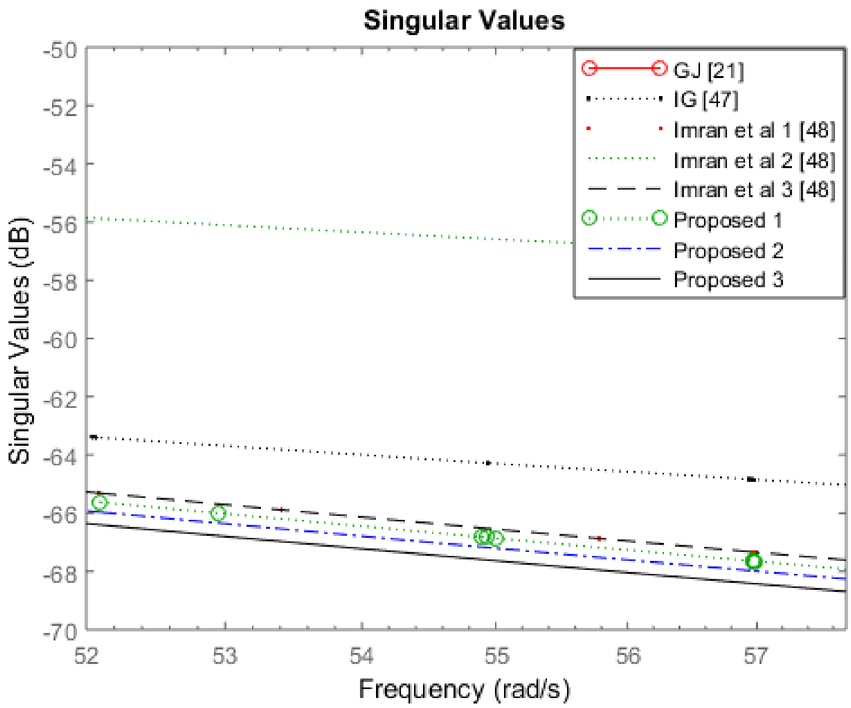

Figure 4.8: Closed view of error plot in desirable frequency interval for example 4.

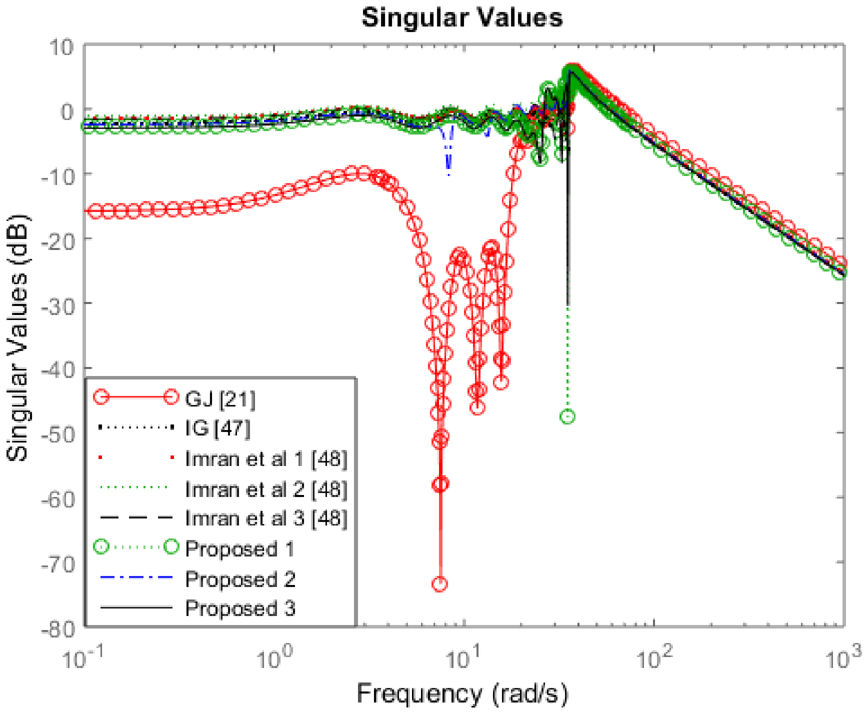

Figure 4.9: Frequency response error with comparison for example 5.

having stop band ripple of 15 and stop band edge frequency of 35Hz with desired frequency interval [5 - 18] rad/s. Fig 4.9 present $\sigma[G(s) - G_r(s)]$, where $G_r(s)$ is of 6^{th} order ROM obtained using GJ [21], IG [47] and Imran et al. [48] and proposed techniques. Fig 4.10 presents the closed view of Fig 4.9 in desirable frequency band. It can be clear from the Fig 4.10 that ROM acquired from proposed techniques mostly have low approximation error when these are compare to prior stability preserving methods IG [47] and Imran et al. [48] in desired frequency band.

Discussion: It can be observed from figures 4.2, 4.4, 4.6, 4.8 and 4.10 that proposed methods mostly have low approximation error when these are compared to other prior stability preserving methods IG [47], Imran et al [48] etc. Since

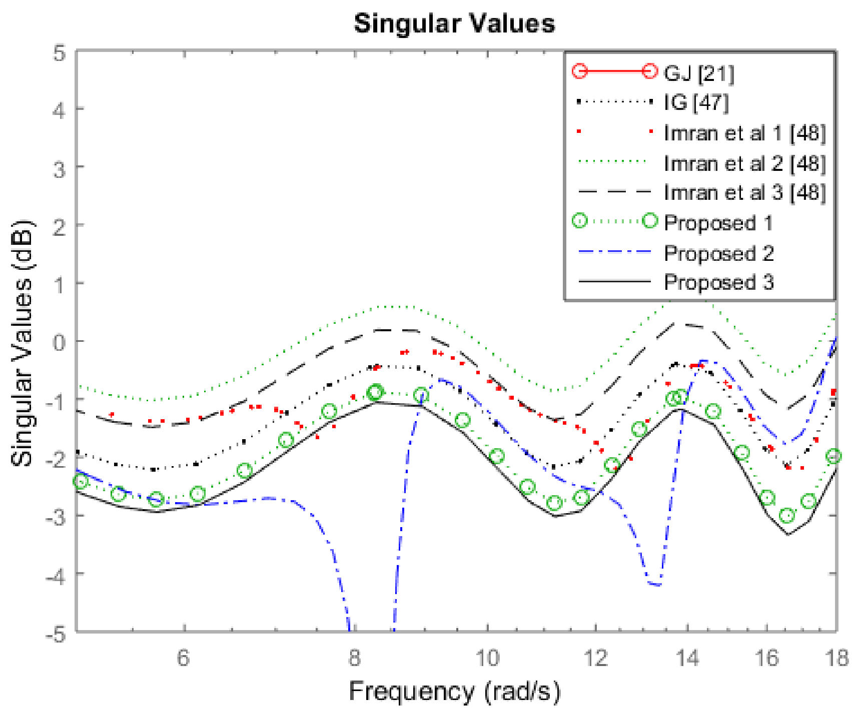

Figure 4.10: Closed view of error plot in desirable frequency range for example 5.

GJ [21] technique yield least approximation error in an cases however, it gives unstable ROM as shown in Table 4.1.

4.2 Examples for Discrete Time Systems Case

This section contains numerical examples showing frequency response error of existing and proposed techniques in the desire frequency range.

Example 6: Suppose a stable 6^{th} order original system with given state space representation:

$$A = \begin{bmatrix} -0.5360 & 1.3029 & 0.2331 & -1.0870 & -0.1404 & 0.2248 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$C = \begin{bmatrix} -0.2085 & -0.6545 & 0.1212 & 0.7666 & -0.1431 & -0.4029 \end{bmatrix}$$

$$D = 0$$

in the desired frequency range $[0.65\pi - 0.8\pi]$ rad/s. Fig 4.11 compares the singular values of frequency response error $\sigma[H(z) - H_r(z)]$, where $H_r(z)$ is 4^{rd} order ROM achieved by using WZ [40], IG [49], Hamid et al. [50] and proposed methods. Fig 4.12 shows the closed view of Fig 4.11 in the desired frequency range. It can be observed from Fig 4.12 that proposed methodss mostly have low approximation error when these are compared with prior stability preserving methods IG [49] and Hamid et al [50] respectively in the desired frequency range.Table II shows the poles of 3^{rd} order ROM obtained by WZ [40], IG [49], Hamid et al [50]

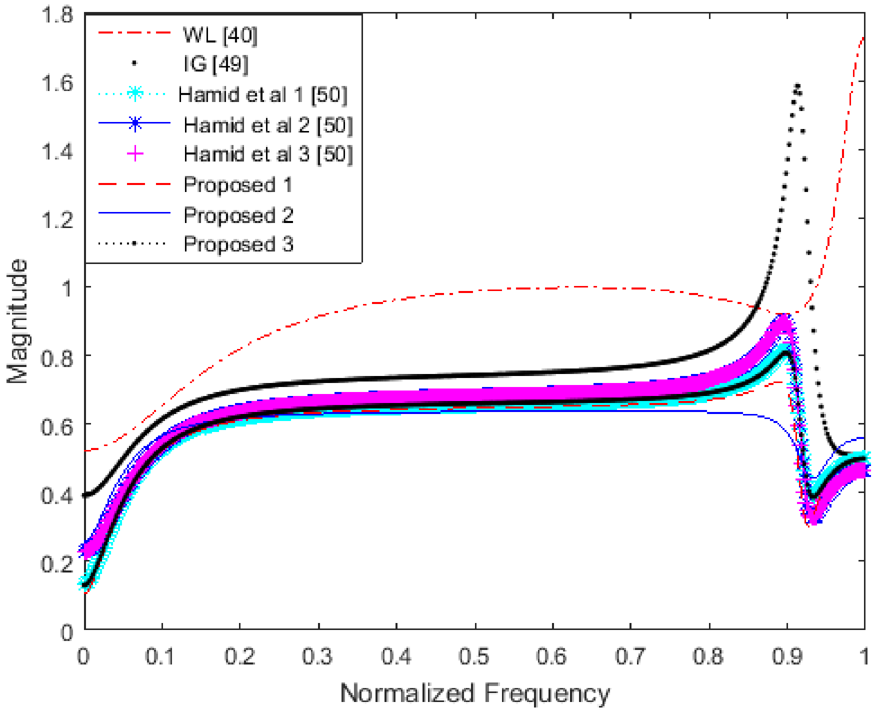

Figure 4.11: Frequency response error with comparison for Example 6

and proposed techniques. It can be seen that WZ [40] yield unstable ROM with pole location at $z = 1.1430$.

Example 7: Consider a stable 8^{th} order original system with following state space representation:

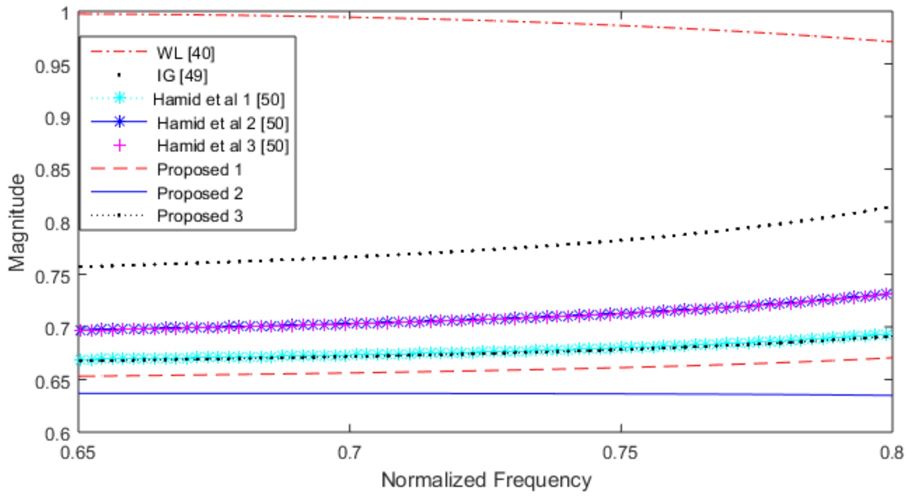

Figure 4.12: Closed view of reduced error in the desirable frequency interval for Example 6

Table 4.2: Roots of 3^{rd} order ROM of EXAMPLE 6

Techniques	Roots
WZ [40]	-1.1430, -0.7834 0.5141
IG [49]	-0.9215 ± 0.2449i, 0.7566
Hamid et al 1 [50]	-0.9143 ± 0.2511i, 0.7807
Hamid et al 2 [50]	-0.9097 ± 0.2590i, 0.7570
Hamid et al 3 [50]	-0.9097 ± 0.2590i, 0.7573
Proposed 1	-0.9127 ± 0.2519i, 0.7867
Proposed 2	-0.8912 ± 0.2052i, 0.8291
Proposed 3	-0.9141 ± 0.2511i, 0.7814

$$A = \begin{bmatrix} 1.17 & -1.87 & 1.75 & -2.386 & 1.69 & -1.483 & 0.7567 & -0.5720 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

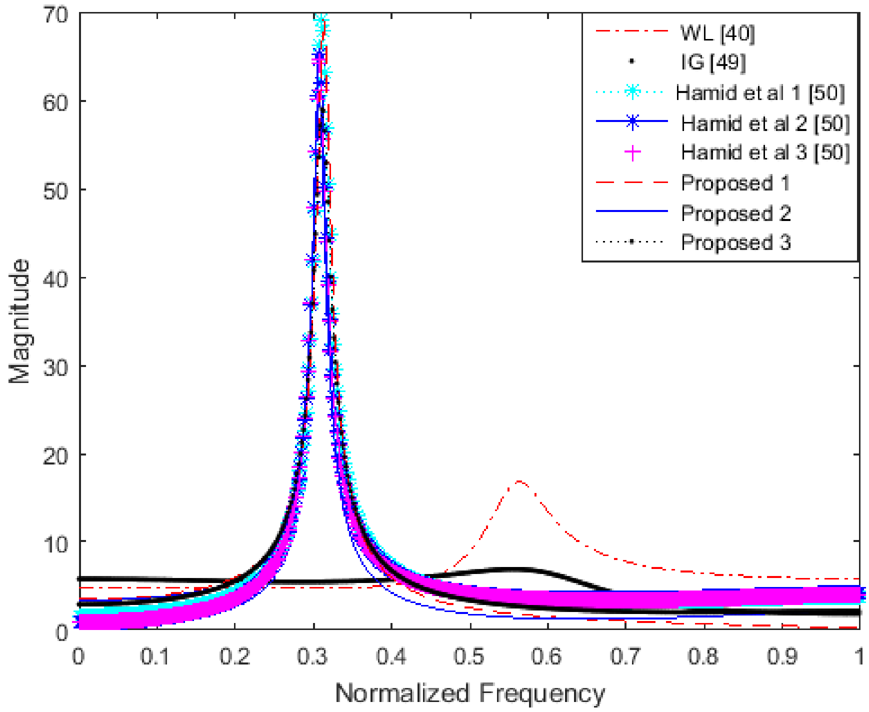

Figure 4.13: Frequency response error with comparison for Example 7

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$
$$C = \begin{bmatrix} 0 & 0 & 0 & 0.256 & 1.74 & 0.45 & 0.9 & 6.97 \end{bmatrix}$$
$$D = 0$$

in a desired frequency range $[0.45\pi - 0.6\pi]$ rad/s. Fig 4.13 shows the comparison of singular values of frequency response error $\sigma[H(z) - H_r(z)]$, where $H_r(z)$ is 3^{rd} order ROM achieved by WZ [40], IG [49], Hamid el at. [50] and proposed methods. Fig 4.14 shows the closed view of Fig 4.13 in the desirable frequency interval. It can be observed from the Fig 4.14 that proposed techniques mostly have low approximation error when these are compared with prior stability preserving methods IG [49] and Hamid et al [50] respectively in the desirable frequency interval.

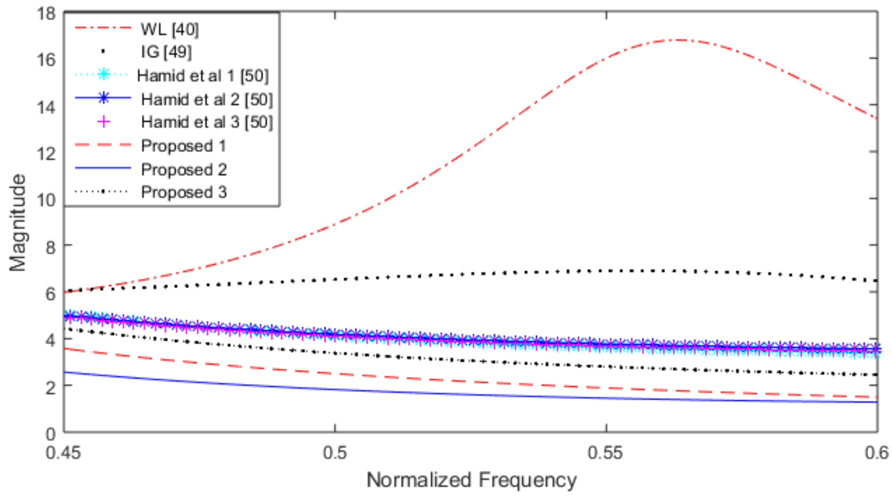

Figure 4.14: Closed view of reduced error in the desirable frequency range for Example 7

Example 8: Let the 18^{th} order, bandpass butterworth stable filter which passes frequencies between 0.4 to 0.8Hz in the desirable frequency range $[0.7\pi - 0.8\pi]$ rad/s. Fig. 4.15 shows the comparison of singular values of frequency response error $\sigma[H(z) - H_r(z)]$, where $H_r(z)$ is of 5^{th} order ROM acquired by WZ [40], IG [49] and Hamid et al. [50] and proposed methods. Fig. 4.16 shows closed view of Fig. 4.15 in the desired frequency band. It can be clear from the Fig. 4.16 that ROM acquired from proposed techniques mostly have low approximation error when these are compared to other prior stability preserving methods IG [49] and Hamid et al. [50] respectively in the desirable frequency range.

Example 9: Let suppose a 30^{th} order chebyshev type 2 stable high pass filter whcih passes frequency between 0.3 to 0.9Hz with 67 dB of ripple in the desired frequency interval $[0.7\pi - 0.85\pi]$ rad/s. Fig. 4.17 shows the comparison of singular values of frequency response error $\sigma[H(z) - H_r(z)]$, where $H_r(s)$ is 8^{th} order ROM obtained using WZ [40], IG [49] and Hamid et al. [50] and proposed meth-

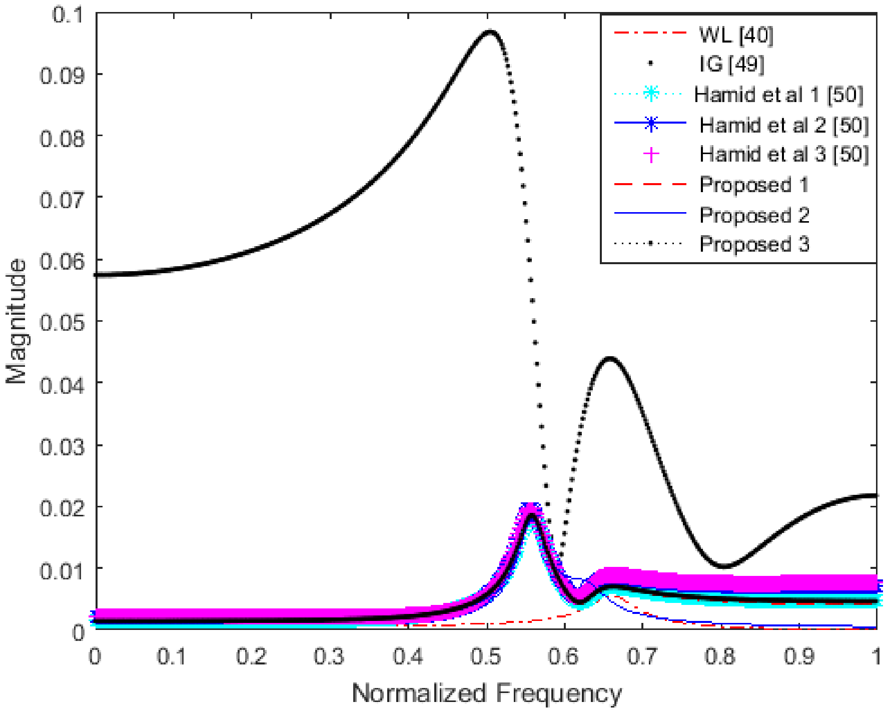

Figure 4.15: Frequency response error with comparison for Example 8

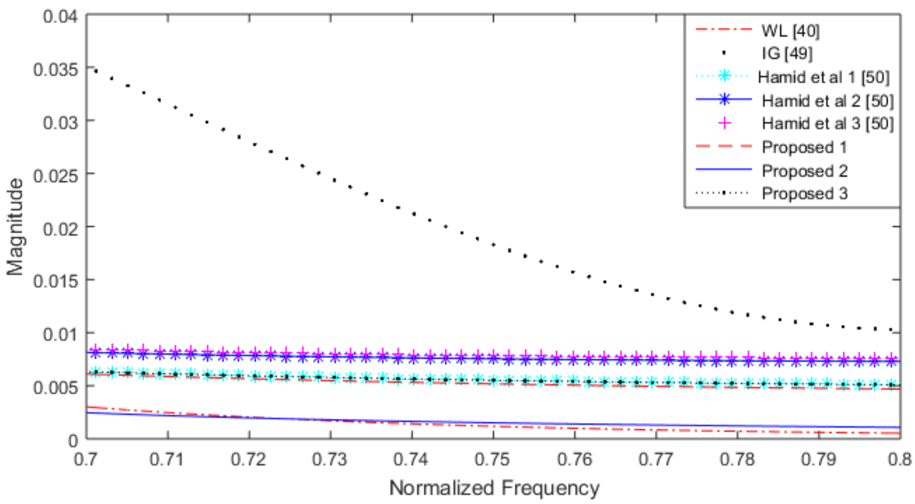

Figure 4.16: Closed view of reduced error in the desirable frequency range for Example 8

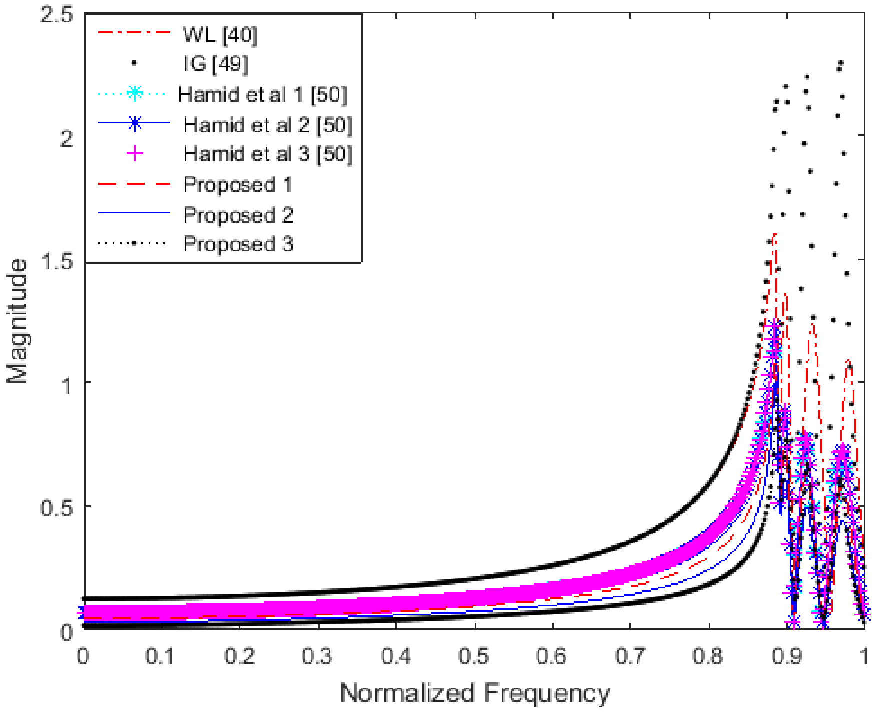

Figure 4.17: Frequency response error with comparison for Example 9

ods. Fig. 4.18 presents the closed view of Fig. 4.17 in desirable frequency band. It can be clear from the Fig. 4.18 that ROM acquired from proposed methods mostly have low approximation error when these are compare to prior stability preserving methods IG [49] and Hamid et al. [50] respectively in desirable frequency range.

Discussion: It can be observed from figures 4.12, 4.14, 4.16 and 4.18 that proposed techniques mostly have low approximation error when compared with other prior stability preserving methods IG [49], Hamid et al [50] etc. Since WZ [40] technique yield least approximation error in an cases however, it gives unstable ROM as shown in Table 4.2.

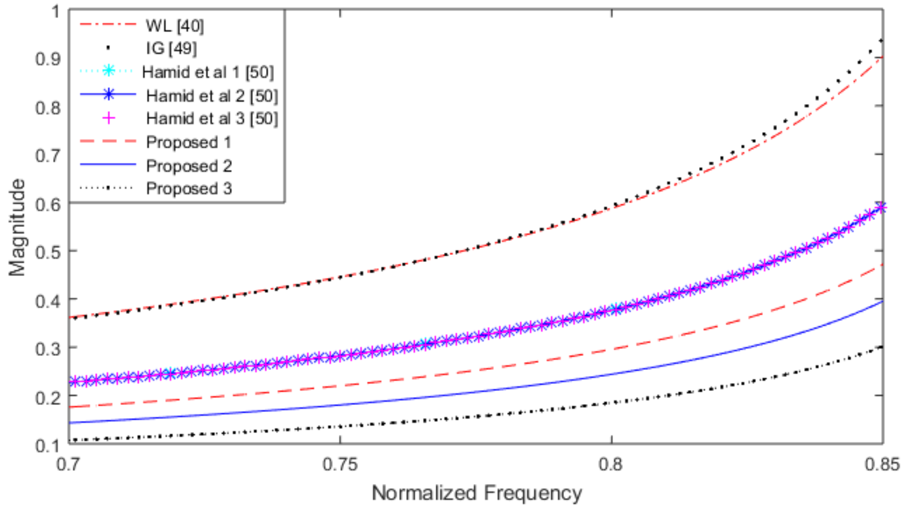

Figure 4.18: Closed view of reduced error in the desirable frequency range for Example 9

Chapter 5

Conclusions and Future Work

5.1 Conclusion

To conclude, distinct limited frequency Gramians based model order reduction methods for both continuous and discrete time systems cases have been presented. Simulation results clarify the effectiveness of proposed techniques by comparison with prior stability persevering methods of model order reduction. Proposed methods have least approximation error in continuous as well as discrete time systems along with carries error bound.

5.2 Future Work

Some novel research fields in the thesis are summarized below: FLMOR methods use BT (chapter 3 and 4). It is exciting to see by using distinct model reduction methods like Pade, Krylov and Hankel norm approximation methods inspire of BT have good results or not. Fictitious input matrix and output matrix has a lot of formulas used in reduction process of various methods. That is not clarify that those of have the well result in cases of lower approximation error and needs further investigation. Proposed methods are only suitable for stable linear time invariant actual system. This is exciting to observe either such results still valid for non-linear and time variant systems.

Bibliography

[1] B. C. Moore,Principal component analysis in linear systems:controllability, observability, and model reduction. IEEETrans. Autom. Control. **26**(1). 17-32 (1981).

[2] K. Glover, "All optimal Hankel-norm approximation of linear multivariable systems and their L1 - error bounds", International Journal of Control, vol. 39, no. 6, pp. 1115-1193, 1984.

[3] D. F. Enns, "Model reduction with balanced realizations: an error bound and a frequency weighted generalization", Proceedings of Conference on Decision and Control, Las Vegas, pp. 127-132, 1984.

[4] C.-A. Lin, and T.-Y. Chiu, "Model reduction via frequency weighted balanced realization", Control-Theory and Advanced Technology, vol. 8, pp. 341-451, 1992.

[5] G.Wang, V. Sreeram, andW. Q. Liu, "A new frequency weighted balanced truncation method and an error bound", IEEE Transactions on Automatic Control, vol. 44, no. 9, pp. 1734-1737, 1999.

[6] P. Vuillemin, C. P. Vassal, and D. Alazard, "A frequency-limited H2 model approximation method with application to a medium-scale flexible air-

craft", Advances in Aerospace Guidance, Navigation and Control, Springer Berlin Heidelberg, pp. 569- 583, 2013. 112

[7] Y. Shamash, "Stable reduced-order models using Pad-type approximations", IEEE Transactions on Automatic Control, vol. 19, pp. 615-619, 1974.

[8] Y. Liu, and B. D. O. Anderson, "Singular perturbation approximation of balanced systems", International Journal of Control, vol. 50, no. 4, pp. 1379-1405, 1989.

[9] Z. Z. Qi, Y. L. Jiang, and Z. H. Xiao, "Model order reduction based on general orthogonal polynomials in the time domain for coupled systems", Journal of the Franklin Institute, vol. 351, no. 6, pp. 3200-3214, 2014.

[10] T. Stykel, "Gramian-based model reduction for descriptor systems", Mathematics of Control Signals and Systems, vol. 16, no. 4, pp. 297-319, 2004.

[11] V. Mehrmann, and T. Stykel, "Balanced truncation model reduction for large-scale systems in descriptor form", Dimension Reduction of Large-Scale Systems, Springer Berlin Heidelberg, pp. 83-115, 2005. 114

[12] P. Aghaee, A. Zilouchian, S. Ravesh, and A. Zadegan, "Principal of frequency-domain balanced structure in linear systems and model reduction", Computer Electronics Engineering, vol. 29, no. 3, pp. 463–477, 2003.

[13] M. Tahavori, and H. R. Shaker, "Model reduction via time-interval balanced stochastic truncation for linear time invariant systems", International Journal of Systems Science, vol. 44, no. 3, pp. 493-501, 2013.

[14] P. Benner, S. Gugercin, and K. Willcox, “A survey of model reduction methods for parametric systems”, MPIMD/13-14, Max Planck Institute Magdeburg Preprint, 2013.

[15] P. Benner, T. Breite, and T. Damm, “Krylov subspace methods for model order reduction of bilinear discrete-time control systems”, Proceedings in Applied Mathematcs and Mechanics, vol. 1, no. 10, pp. 601-602, 2010.

[16] A. Davoudi, J. Jatskevich, P. L. Chapman, and A. Bidram, “Multi-resolution modeling of power electronics circuits using model-order reduction techniques”, IEEE Transactions on Circuits and Systems I, Regular Papers, vol. 60, no. 3, pp. 810-823, 2013.

[17] X. Du and G. H Yang, “H1 model reduction of linear continuous-time systems over finite-frequency interval”, IET Control Theory and Applications, vol. 4, no. 3, pp. 499-508, 2010.

[18] W. Yang, L. Zhang, P. Shi and Y. Zhu, “Model reduction for a class of non-stationary Markov jump linear systems”, Journal of the Franklin Institute, vol. 349, no. 7, pp. 2445-2460, 2012 .

[19] M. B. Ha, C. Batlle and E. Fossas, “A new estimation of the lower error bound in balanced truncation method”, Automatica, vol. 50, no. 8, pp.2196-2198, 2014.

[20] H. R. Shaker and M. Tahavori, “Time interval model reduction of Bilinear systems”, International Journal of Control, vol. 87, no. 8, pp. 1487-1495, 2014.

[21] W. Gawronski, and J.-N. Juang, “Model reduction in limited time and frequency intervals”, International Journal of System Sciences, vol. 21, no. 2, pp. 349-376, 1990.

[22] B. Besselink, N. V. Wouw, J. M. A. Scherpen and H. Nijmeijer "Model reduction for nonlinear systems by incremental balanced truncation", IEEE Transactions on Automatic Control, vol. 59, no. 10, pp. 2739-2753, 2014.

[23] Z. Bai, and R. W. Freund, "A partial Pade-via-Lanczos method for reduced-order modeling", Linear Algebra Applications, vol. 332, no. 1, pp. 139-164, 2001.

[24] S. Gugercin, "An iterative SVD-Krylov based method for model reduction of largescale dynamical systems", Linear Algebra Applications, vol. 428, no. 8, pp. 1964- 1986, 2008.

[25] A. Ghafoor, and V. Sreeram, "Partial-fraction expansion based frequency weighted model reduction technique with error bounds", IEEE Transactions on Automatic Control, vol. 52, no. 10, pp. 1942-1948, 2007.

[26] A. Ghafoor, and V. Sreeram, "A survey/review of frequency-weighted balanced model reduction techniques", Journal of Dynamic Systems, Measurement and Control, vol. 130, no. 6, 2008.

[27] M. Imran and A. Ghafoor, "Stability preserving model order reduction technique using frequency limited Gramians for discrete time systems ", IEEE Transactions on Circuits and Systems II: Express Briefs, vol. 61, no. 9, pp. 716-720, 2014.

[28] M. Imran, A. Ghafoor, and V. Sreeram, "Frequency weighted model order reduction technique and error bound", Automatica, In Press, 2014.

[29] M. Imran, A. Ghafoor, and V. Sreeram, "Limited frequency interval Gramians based model reduction for Generalized non-singular discrete time systems", IET Control Theory and Applications, Accepted with revisions, 2014.

[30] M. Imran and A. Ghafoor, "Limited frequency Gramians based model reduction for Generalized non-singular systems ", IMA Journal of Mathematics, Control and Information, In Press, 2014.

[31] M. Imran and A. Ghafoor, "Model reduction of Descriptor systems using frequency limited Gramians ", Journal of The Franklin Institute, In Press, 2014.

[32] M. Imran and A. Ghafoor, "Limited frequency Gramians based model reduction technique and error bound ", Circuits, Systems and Signal Processing, Submitted, 2014.

[33] V. Sreeram, "A new frequency weighted balanced related technique with error bound", Proceedings of Conference on Decision and Control, Bahamas, pp. 3084-3089, 2004.

[34] V. Sreeram, B. D. O. Anderson, and A. G. Madievski, "New results on frequency weighted balanced reduction technique", Proceedings of American Control Conference, Seattle, pp. 4004-4009, 1995.

[35] V. Sreeram, and P. Agathoklis, "Discrete system reduction via impulse-response Gramians and its relation to q-Markov covers", IEEE Transactions on Automatic Control, vol. 37, no. 5, pp. 653-658, 1992.

[36] A. Varga, and B. D. O. Anderson, "Accuracy-enhancing methods for balancingrelated frequency-weighted model and controller reduction", Automatica, vol. 39, no. 5, pp. 919-927, 2003.

[37] H. I. Nurdin, "Structures and transformations for model reduction of linear quantum stochastic systems", IEEE Transactions on Automatic Control, vol. 59, no. 9, pp. 2413-2425, 2014.

[38] S. Gugercin and A. C. Antoulas, "A survey of model reduction by balanced truncation and some new results", International Journal of Control, vol. 77, no. 8, pp. 748-766, 2004.

[39] U. M. Al-Saggaf, and G. F. Franklin, "On model reduction", Proceedings of Conference on Decision and Control, pp. 1064-1069, 1986.

[40] D.Wang, and A. Zilouchian, "Model reduction of discrete linear syetem via frequency domain balanced realization', IEEE Transactions on Circuits and Systems I, Fundamental Theory and Applications, vol. 47, no. 6, pp. 830-837, 2000.

[41] K. Zhou, "Frequency weighted L1 norm optimal Hankel norm model reduction", IEEE Transaction on Automatic Control, vol. 40, no. , pp. 1687-1699, 1995.

[42] S. Sahlan, and V. Sreeram, "New results on partial fraction expansion based frequency weighted balanced truncation", Proceedings of American Control Confernce, St. Loius, pp. 5695-5700, 2009.

[43] V. Sreeram, S. Sahlan, "Improved results on frequency weighted balanced truncation and error bounds", International Journal of Robust and Nonlinear Control, vol. 22, no. 11, pp. 1195-1211, 2012.

[44] G. A. Latham and B.D.O. Anderson, "Frequency weighted optimal Hankel-norm approximation of stable transfer function", Systems and Control Letters, vol. 5, no. 4, pp. 229-236, 1986.

[45] Y. S. Hung and K. Glover, "Optimal Hankel-norm approximation of stable systems with first-order stable weighting functions", System and Control Letters, vol. 7, no. 3, pp. 165-172, 1986.

[46] U. M. Al-Saggaf, and G. F. Franklin, “Model reduction via balanced realizations: an extension and frequency weighting techniques”, IEEE Transaction on Automatic Control, vol. 33, no. 7, pp. 687-692, 1988. 115

[47] M. Imran and A. Ghafoor, “A frequency limited interval Gramians based model reduction technique with error bounds,” Circuit Syst. Signal Process., vol. 34, no. 11, pp. 3505–3519, (2015).

[48] Imran, Muhammad, and Abdul Ghafoor. ”Frequency limited model reduction techniques with error bounds.” IEEE Transactions on Circuits and Systems II: Express Briefs 65.1: 86-90 (2017).

[49] Imran, Muhammad, and Abdul Ghafoor. ”Stability preserving model reduction technique and error bounds using frequency-limited Gramians for discrete-time systems.” IEEE Transactions on Circuits and Systems II: Express Briefs 61.9 (2014): 716-720.

[50] Toor, Hamid Imtiaz, et al. ”Frequency Limited Model Reduction Techniques for Discrete-Time Systems.” IEEE Transactions on Circuits and Systems II: Express Briefs (2019).