CONTROL DESIGN OF APPROXIMATED SYSTEMS USING FEEDBACK ANALYSIS OF STATES AND OUTPUT

By

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ABSTRACT

Scientific demonstration is a fundamental component for the investigation and outline of a dynamical systems. For the most part, extensive and complex models are acquired from physical systems. A few illustrations are automated broadcast communications, mechanical and numerous other complex systems. These systems are administered by the fractional differential, Laplace and integro-differential equations and so forth. For the investigation and plan of such systems, diminished request models are alluring that give a decent estimation of the original systems. In most recent couple of decades, remarkable exploratory work has been done on various parts of approximation of original systems. Existing techniques of approximation of original systems are having some limitations to perform the approximation of systems and obtain the stable approximated systems. New techniques are proposed that reduce these 1-D systems into their reduced order form. The proposed technique ensures the stability of the reduced order system and also provides the low approximation error as compared to other existing stability preserving techniques. This thesis is also fulfilling the limitation of previous 1-D Lower Order Approximation Systems techniques incase of continuous and discrete Gramians based Lower Order Approximation Systems. Simulation results show the effectiveness of the proposed transformation along with 1-D stability preserving technique. This thesis is also fulfilling the instability issue of lower order approximation of original systems by introducing different algorithms, static state feedback controller of lower order approximated systems incase continuous time systems, static observer based state feedback controller for continuous time systems, static state feedback controller of lower order approximated systems incase discrete time systems, static observer based state feedback controller for discrete time systems. Lower Order Approximation Systems algorithms primarily based on spectral projection strategies are composed of primary matrix computations such as fixing linear systems, matrix products, and QR factorizations. The use of these libraries enhances both the reliability and portability of the Lower Order Approximation routines. The performance will depend on the efficiency of the underlying serial and parallel computational linear algebra libraries and the verbal exchange routines. In this thesis, control design of approximated models along with different examples among different techniques are presented which shows the effectiveness of the proposed techniques.

DEDICATION

Dedicated to my mother Captain Rehmat Noor, my sister Colonel Huma Nasir and my son Taha for their endless love and support

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It brings me fathomless joy to have first journal of my life, how this concept became a reality is prodigious in terms of its efforts and deity. Certainly, the biggest influence with respect to background, motivation direction and profound help has been from Muhammad Imran, who has been a great mentor in his talks and actions. Not only this, he has proven to be a brother, a friend, a guide and beyond which many relationships can be labelled to his accounts. With him, equal efforts are credited to my colloquial sister, Sammana Batool, who has been a page turner of my life. Her efforts for the completion of this paper and putting it to reality are phenomenal. No words can describe the gratitude.

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NOTATION

Transfer function $G(s)$	=	$C(sI-A)^{-1}B+D \Leftrightarrow$ state-space realization $\{A, B, C, D\}$
$\\|G(jw)\\|_\infty$	=	$\sup_\omega \bar{\sigma}(G(j\omega))$, if $G(jw)$ is a transfer function (matrix) where $\bar{\sigma}(G(j\omega))$ is the maximum singular value of $G(jw)$
$P>0$	=	Positive definite matrix P Symmetric matrix P with positive eigenvalues
$P\geq 0$	=	Positive semidefinite matrix P Symmetric matrix P with non-negative eigenvalues
X^T	=	Transpose of matrix or vector X
X^*	=	Complex conjugate transpose of matrix or vector X
$X^\dagger$	=	Pseudoinverse of matrix or vector X
X^{-1}	=	Inverse of matrix X
$\lambda_i[X]$	=	Eigenvalues of X
$\|X\|$	=	Modulus of X
$\mathrm{diag}(\Sigma_1, \Sigma_1)$	=	$\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$
$\Sigma_{i=1}^n \sigma_i$	=	$\sigma_1+\sigma_2+\cdots+\sigma_n$
$\Pi_{i=1}^n \sigma_i$	=	$\sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_n$

ACRONYMS

Lower Order Approximated Systems	LOAS
Frequency Limited Lower Order Approximated Systems	FLLOAS
Impulse Response Gramian	IRG
Frequency Limited Impulse Response Gramian	FLIRG
Infinite Impulse Response	IIR
Discrete Fourier transform	DFT
Discrete Time Systems	DTS
Continous Time Systems	CTS

Chapter 1

INTRODUCTION

1.1 Overview

Control is employed to exchange the conduct of a model for this purpose it behaves in a special way over time. For example, we have a tendency to would maybe select the tempo of a car on the highway to continue to be as shut as conceivable to sixty miles per hour in spite of workable hills or unsafe wind; or we have a tendency to may additionally choose an craft to comply with a liked altitude, heading, and tempo profile unbiased of wind gusts; or we have a tendency to would possibly favor the temperature and stress in an pretty reactor vessel in an in the main natural motion plant to be maintained at liked levels of these are being achieved these days by control methods and as a result the higher than are examples of what computerized control constructions are designed to attempt to do, barring human intervention control is used every time portions like speed, altitude, temperature, or voltage want to be created to behave in some proper over time.

To reap some grasp into how an computerized system operates we shall rapidly appear at the temporal control mechanism at some point of a automotive. It is per chance instructive to suppose about first how a ordinary driver ought to control the car speed over uneven terrain. The driver, with the aid of fastidiously staring at the meter, and exactly developing or reducing the gas flow to the engine, exploitation the throttle, will maintain the speed quite accurately. Higher accuracy will possibly be carried out with the aid of capability. An automated temporal system, additionally referred to as cruise control, works with the resource of means of exploitation the difference, or error, between the unique and preferred speeds and data of the car's response to gas will lengthen and reduces to calculate through means of the use of some algorithmic rule an appropriate treadle position, therefore to energy of the steady state error to zero. This name method is named an affect guidelines and it's enforced in the controller. The device configuration is tested in Fig. 1.1. The vehicle dynamics of interest are captured in the plant, data involving the unique output is fed to the controller with the useful resource of sensors, and subsequently the manage choices are enforced by means of way of the usage of a tool, the mechanism, that changes the function of the fuel pedal. The information of the car's response to fuel will enlarge and reduces is most in many instances captured at some factor of a mathematical model. Certainly in an auto at present day there are larger computerized control structures like the anti-lock brake machine (ABS) and emission control. The utilization of remarks control preceded manipulate theory, outlined inside the following sections, thru over 2000 years. The predominant remarks system on file is that the widespread Water Clock of Ktesibios in Alexandria, Egypt, from the 3^{rd} century BC.

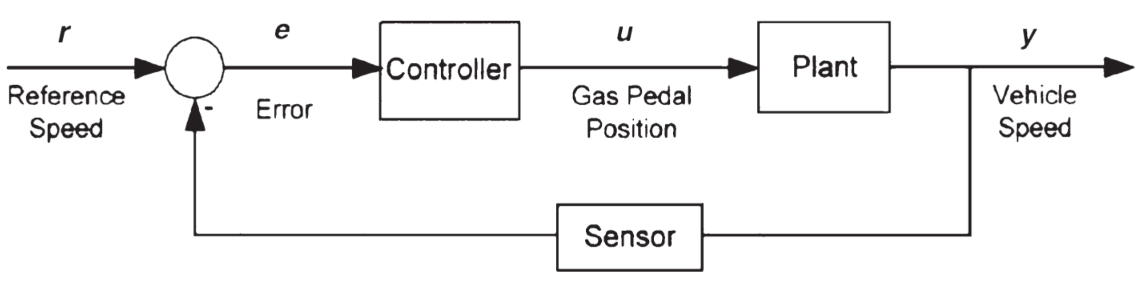

Figure 1.1: Feedback Control Model.

To diagram a controller that produces a device behave in the direction of a appropriate manner, we desire some way to predict the habits of the parts of interest over time, specifically how they alter in response to virtually one of a sort inputs. Mathematical fashions are most regularly used to predict future behavior, and gadget plan methodologies are supported such models, critical mathematical thoughts and skills, like fixing differential equations and the use of laplace transform. The role of manage concept is to help us reap insight on how and why remarks control constructions work and how to systematically deal with countless plan and contrast problems. Specifically, the following troubles are of every smart magnitude and theoretical interest:

The Role of Control Theory

1. Stability and stability margins of closed-loop systems.

2. how speedy and convenient the error between the output and the point is pushed to zero.

3. how well the gadget handles surprising external disturbances, sensor noises, and interior dynamic changes.

In the following, modeling and assessment are first introduced, followed via way of an summary of the classical layout strategies for single-input single-output plants, sketch evaluation methods, and implementation problems. Alternative structure techniques are then temporarily bestowed. Finally, For the sake of simplicity and brevity, the discussion is confined to linear, time invariant systems. Results per chance located inside the literature for the cases of linear, time-varying systems, and conjointly for nonlinear systems, systems with delays, constructions represented by using partial differential equations, then on; these results, however, have a tendency to be extra constrained and case dependent.

The proportional-integral-derivative (PID) controller, described through

$$u = K_P e + K_I \int e + K_D \dot{e} \tag{1.1}$$

is a significantly beneficial manipulate approach that was invented over eighty years past. Here K_P, K_I and K_D are controller parameters to be elect, normally with the aid of trial and error or via way of the use of a search table in alternate apply. The goal, as internal the controller example, the energy of the steady state error to zero at some point of a best manner. All three phrases equation. 1.1 have specific bodily meanings in this $\boldsymbol{e}$ is that the modern day error, $\int \boldsymbol{e}$ is that the amassed error, and $\dot{e}$ represents the trend. This, alongside aspect the integral grasp of the causative relationship between the manage signal **(u)** and additionally the output **(y)**, varieties the premise for engineers to “tune”, or alter, the controller parameters to fulfill the seem to be specifications. This intuitive style, due to the fact it seems, is spare for a quantity of control applications. To this day, PID control continues to be the predominant strategy in trade and is discovered in over 95 proportion of industrial applications. Its success can additionally be attributed to the simplicity, efficiency, and effectiveness of this technique.

1.2 Low Dimensional Approximated Systems

The derivation of a sensible numerical model is central to get a decent comprehension of the dynamical conduct of physical systems being referred to or to control its behaviour keeping in mind the end goal to accomplish required execution determinations. Reduction of complex frameworks, (for example, chip outline, liquid stream, mechanical systems reproduction) yields substantial scale systems. In spite of the progression of innovation and the regularly expanding computational speed, the investigation, control and enhancement of substantial scale frameworks is testing (if not unthinkable), because of costly calculations and capacity prerequisites. In this manner, procedure of creating a low-dimensional or lower or-

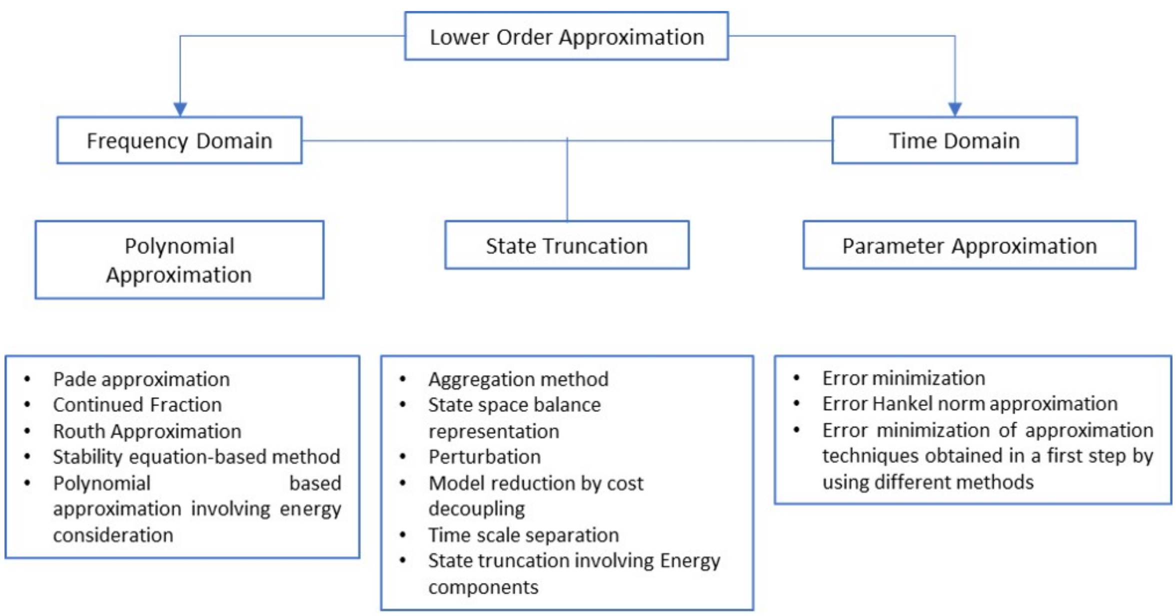

Figure 1.2: Various Schemes for Lower Order Approximation.

der approximation of systems that gives a decent gauge of the original higher order model is known as approximated systems various schemes are organized in Fig. 1.2 . By and large, the point of approximated systems is to discover lower order approximation of systems which approximate the information content of the original systems. This is accomplished with a lower storage requirements and additional assessment time. Approximated systems has assumed a critical part in current control systems research and got ample consideration in the most recent couple of decades. One of the critical factor of approximated systems is the reduction of the error which arises from the distinction between the original and lower order approximation of systems frequency response. Furthermore, the system properties like stability, input-output behaviour and frequency response error bound are also of main concern during the process of approximated systems. Numerical properties, for example, computational speed and precision, storage requirements and so on assume an imperative part in computational effectiveness of the approximated systems procedures. The error bound equation for approximated systems procedure gives the approximation error. It will assist the designer to choose approximated structures machine for the involved software [1]- [21]. We proposed to utilize a rule-based technique to remedy this issue. To this purpose, an growth of statistics base is in increase so that after the desire of approximation algorithm, a appear for the fundamental splendid validation standards will be performed. Moreover, in the match that the divulge approval comes about are not regarded satisfactory, expert procedure may additionally offer assist the purchaser to re-configure the via and massive lessening bother following the cyclic plot represented in Fig.1.3. One of the most widely used approximated systems methods is balanced truncation (BT) [22]. Moore [22], introduced the balanced truncation procedure to perform the approximated systems, it also has a error bound formula. A first arrangement of the open-loop balancing algorithm applied is reported in Fig.1.4. In addition global parallelization method has been carried out, resulting in a second scheme, proven in Fig. 1.5. However, it uses full frequency range for the approximated systems thats why its encourages to introduce the frequency weights. Enns [23] procedure was augmentation of BT [22] to introduction of frequency weights. These assigned weights are valuable for the frequency value shaping of the approximated systems error. Enns [23] technique may utilize input weighting, output weighting or both. In any case, for uneven

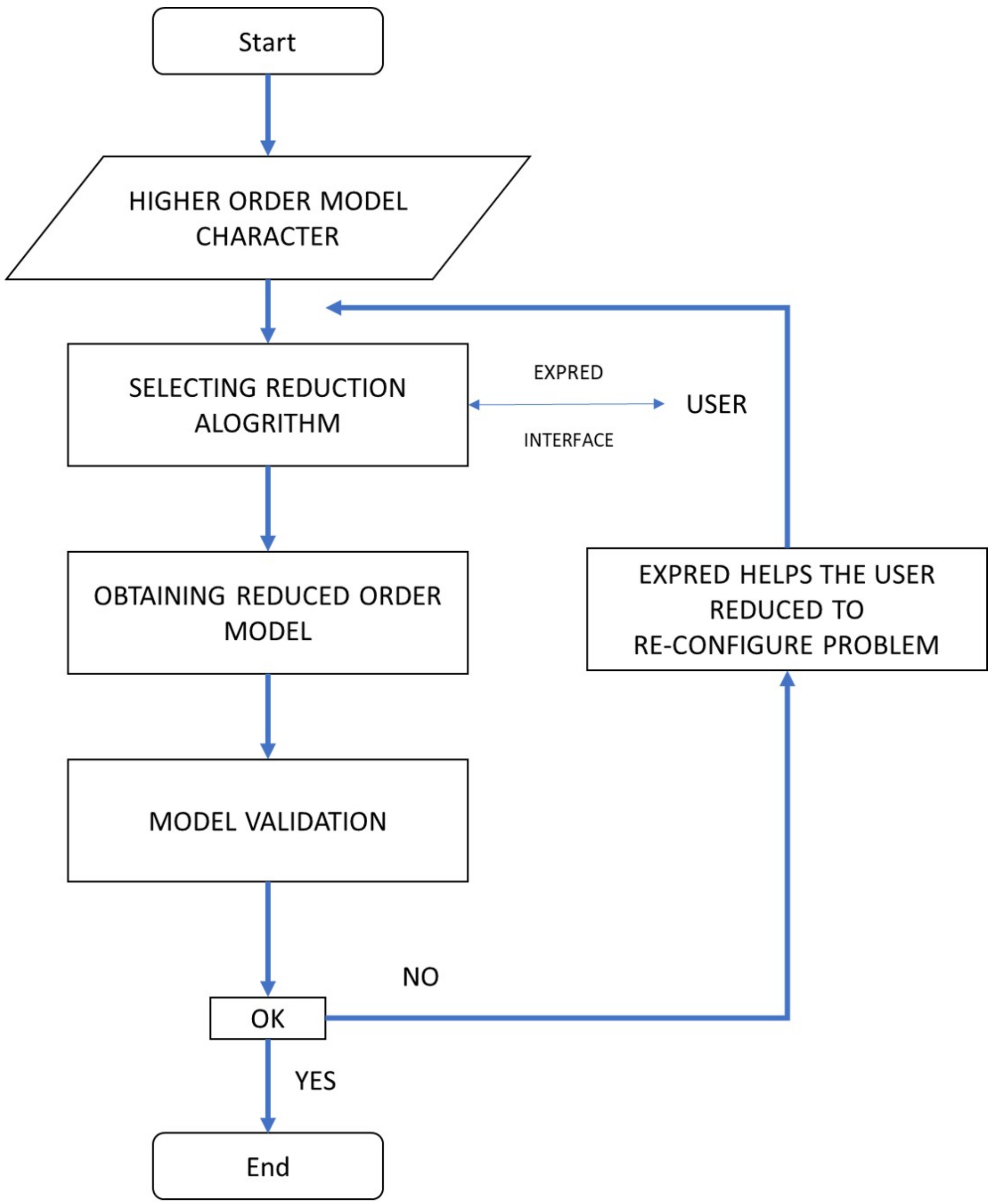

Figure 1.3: Expert procedure lower order approximation flow-chart showing the mode cyclic operation.

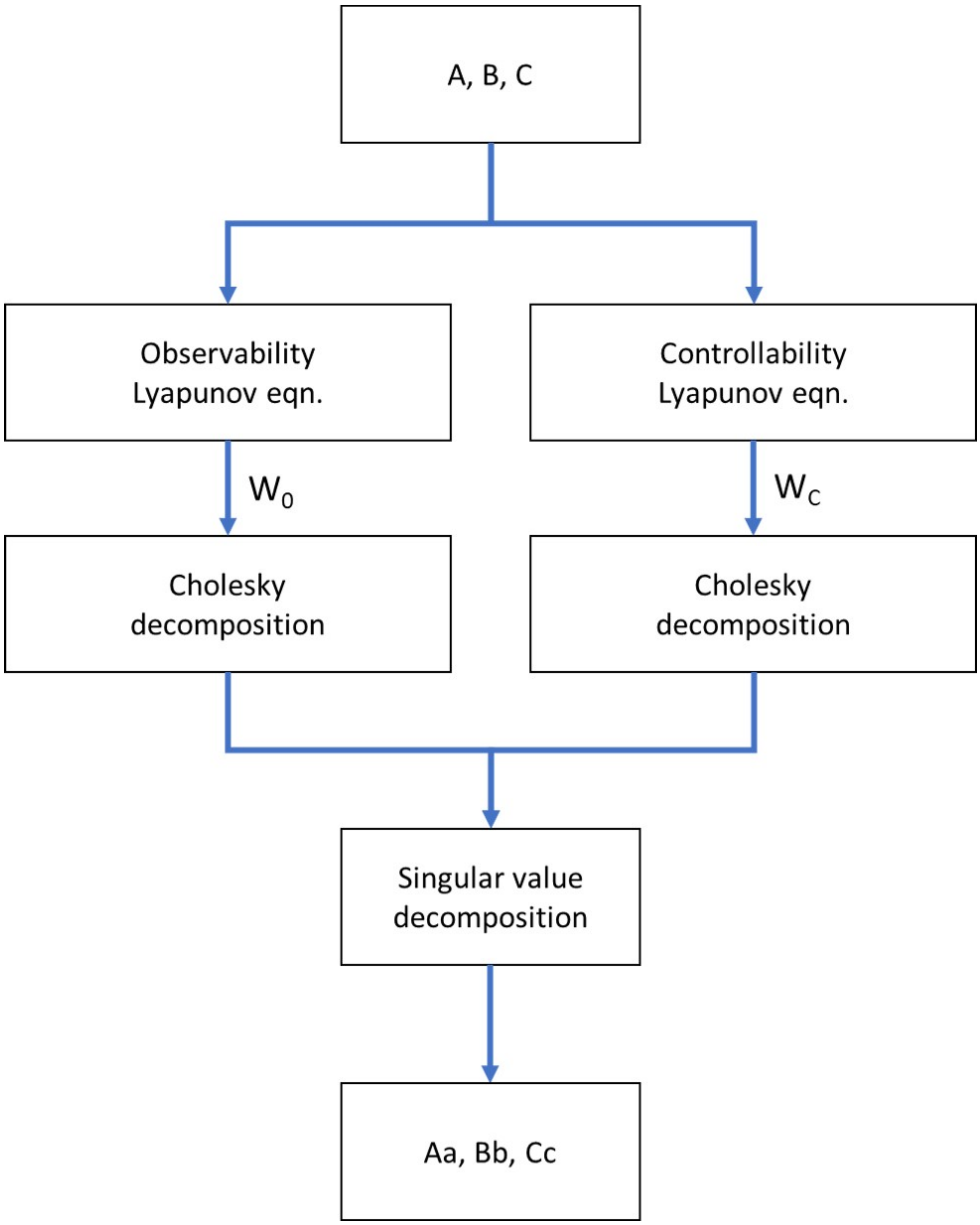

Figure 1.4: Open-loop balancing method.

weighting or stability of lower order approximation of systems is ensured. However, for both sided weighting case, stability is not ensured. To defeat this insecurity issue of both sided weighting, a few adjustments to Enns [23] technique have been suggested. To conquer Enns [23] downside, Lin and Chiu [24] have proposed an alternate strategy that ensures stability when both sided weighting is available. In any case, their procedure can work just when the weighting function used here can be strictly proper and there occurs non poler zero cancellation while framing the augmented systems. These confinements of Lin and Chiu [24] procedure were later adjusted by Sreeram etal [25] and Varga and Anderson [26], which summed up to incorporate proper weights, while holds the stability of the lower order approximation of systems even at the point when poles zero cancelation happen. Be that as it may, Varga and Anderson [26] procedure produces an indistinguishable outcome from Enns [23], espacially in appliances involving controller reduction. So for controller reduction issue, if Enns [23] procedure yields unstable lower order approximation of systems, aso by Varga and Anderson [26] technique. Wang et al's [27] technique also throws light on the instability issue of Enns [23], thus not just give stable lower order approximation of systems within the sight of both sided weighting additionally provide a priori error bound. The error of approximation by Wang et al [27] was later modified by Varga and Anderson [26]. As indicated by Sreeram [28], this strategy (and modification by Varga and Anderson) are realization dependent. This implies that for identical systems, distinctive models can be derived and obtained from various realizations. Another forms of techniques was proposed based on partial fraction, Latham and Anderson [29]. Several frequency weighted model reduction (FWMR) techniques in view of partial fraction expansion idea have been taken after [30]- [34]. Error bounds exist for certain exceptional sort of weighting function. Victor Sreeram [35] technique gives low approximation error but have no theocratical justification. in Fig.1.4. Sahlan and Sreeram [36], in spite of the fact that gives low approximation error

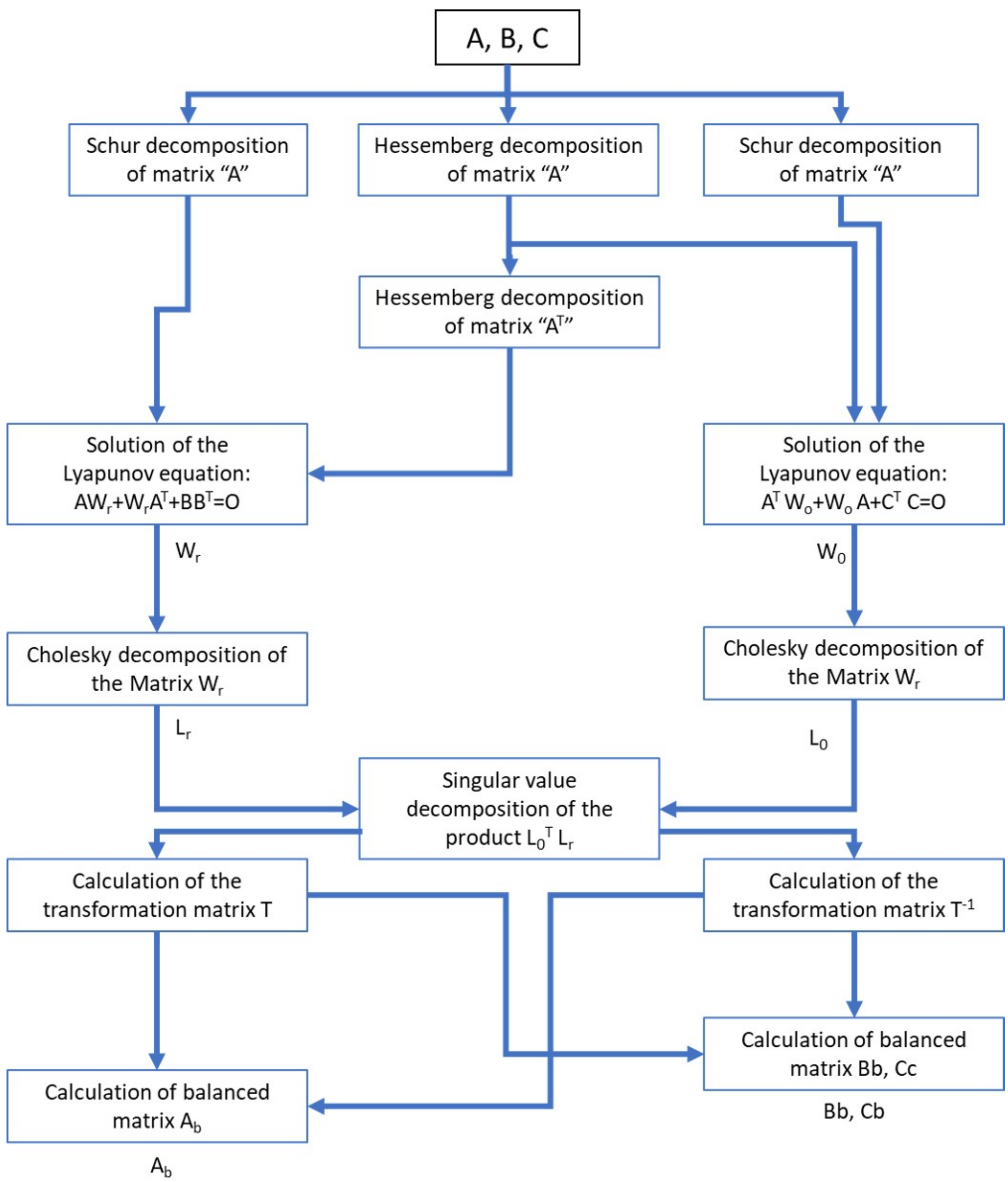

Figure 1.5: Balancing parallel method of Schur decomposition.

when contrasted with Enns [23] technique, and other surely understood FWMR procedures, yet this technique is realization dependent. This idea of FWMR encourages introduction of limited frequency range during the process of approximated systems. Later on, Gawronski and Jaung (GJ) [38] announced the frequency limited Gramians based approximated systems technique for the continuous time systems. However, this technique also has some limitations regarding its stability. The introduced Gramians sometimes produce negative definiteness that may cause unstable . This limitation was covered by Gugercin and Anaulos (GA) [39]and Victor Sreeram (VS) [35]. GA [39] ensured the stability by introducing the square root related to eigenvalues of input-output related matrices. VS [35] ensured the stability, by picking up only positive eigenvalues and ignoring negative eigenvalues of input output related matrices. Recently, Imran [40] proposed a new way to ensure the stability of lower order approximation of systems by choosing circular way of multiplication, division and negative of averaging. This technique also has a drawback that the new diagonal matrix will have last eigenvalue as zero, nullifying the effect of last eigenvalue that may cause a huge variation in original system. These techniques also provides an error bound formula. For the discrete time systems Wang and Zilouchian (WZ) [41] proposed a new technique incase of limited frequency Gramians. However, this technique also may cause unstable lower order approximation of systems. The modification is carried out and new way is proposed to ensure the stability of lower order approximation of systems for the discrete time systems. Victor Sreeram (VS) [42] and Imran [43] proposed a new way to ensure the stability of lower order approximation of systems. VS [42] ensured the stability by introducing two algorithms, in first algorithm the stability is ensured by taking absolute of the all the eigenvalues of some input-output matrices. Whereas, in second algorithm the negative eigenvalues are ignored and only taking the positive eigenvalues of some input output matrices to ensure the stability. Later on, Imran [43] proposed a new way to ensure the stability of the lower order approximation of systemsby taking the least eigenvalue and subtracting it from all the eigenvalues from some input output related matrices to ensure the stability of lower order approximation of systems. However, this technique may cause larger variation as compared other techniques. This technique is nullifying the effect of last eigenvalues that may cause larger variation. The proposed technique also provides the error bound formula. Approximated systems for 1-D systems providing good sight for researchers.

1.3 Feedback Control Design of Low Dimensional Approximated Systems

The state of a dynamical device may want to be a combination of variables that lets in prediction on further future improvement of a system. During this Thesis we are going to explore the thought of dominant a gadget thru state feedback. We count on that the device required to be managed is delineate via a linear state model and accommodates a single input. The comments manipulate are developed step with the aid of step victimisation one single idea: the positioning of control system eigenvalues at desired locations. It seems that the controller comprises a terribly mesmerising shape that applies to a number of sketch methods. This chapter be considered as an example of techniques of many analytical designs. If the nation of a machine is no longer available for direct measurement, it is often practicable to see the state by way of means of reasoning related to the kingdom via our statistics of the dynamics and a lot of constrained measurements. This is done using or constructing an "observer" that makes use of measurements of the inputs and outputs of a linear system, alongside with a mannequin of the device dynamics, to estimate the state. The essential factors of the assessment and designs throughout this chapter are carried out for structures with one enter and one output, however, it turns out that the structure of the controller and additionally the varieties of the equations are exactly the equal for structures with many inputs and masses of outputs. There also are severa choice sketch techniques that provide controllers with equal structure. A attribute attribute of a controller with kingdom feedback and an observer is that the first-class of the controller is given through the complexity of the gadget to be controlled. So the controller genuinely carries a model of the system. This is an instance of the indoors model principle that says that a controller ought to have an internal mannequin of the controlled system.

Enns [23] technique may utilize input weighting, output weighting or both. In any case, for uneven weighting, stability of lower order approximation of systems is ensured. However, for both sided weighting case, stability is not ensured. Gawronski and Jaung (GJ) [38] highlighted the frequency limited Gramians based approximated systems technique for the continuous time systems. However, this technique also has some limitations regarding its stability. The introduced Gramians sometimes produce negative definiteness that may cause unstable.

For the discrete time systems Wang and Zilouchian (WZ) [41] proposed a new technique incase of limited frequency Gramians. However, this technique also may cause unstable lower order approximation of systems.

To defeat this insecurity issue of Enns [23], Gawronski and Jaung (GJ) [38] and Wang and

Zilouchian (WZ) [41] we proposed a new scheme that ensure the stability of lower order approximation of systems by design a feedback controller. That not only ensure the stability of the lower order approximation of systems but produces low error and error bound as compared to existing stability preserving techniques [24], [25], [26], [26], [28], [29], [39], [35], [40], [42], [43].

Chapter 2

CONTROL DESIGN OF APPROXIMATED SYSTEMS USING FEEDBACK ANALYSIS OF STATES AND OUTPUT FOR DISCRETE TIME SYSTEMS

The process of Lower Order Approximation System (LOAS) is to reduce a system from higher order to its lower order for ease in simulation, analysis and design of complex systems, filters and controller [11]- [14]. Balance truncation [22] is a common and useful scheme to get stable LOAS for stable original system. Moreover the scheme also has error bounds. However it uses full frequency range to get LOAS. This encourages to introduce the frequency weights to perform LOAS. Enns [23] extended the work of balance truncation technique to incorporate frequency weights. Enns [23] method may use single sided (input/output) and double sided weights. It yields stable LOAS when use only one side weights whereas with two sided weights, LOAS is unstable. To overcome the problem of Enns, many other techniques are given in literature [25]- [37]. In some cases, a specific range of frequency can be of interest. Wang and Zilouchian (WZ) [41] proposed a frequency limited technique without explicit weights. It can yield unstable LOAS and no error bound exist. To overcome the problem of WZ's [41], Victor Sreeram (GS) [42] proposed two methods which ensures and guarantee the LOAS stability. The work in [42] guarantees stability of LOAS and carry error bounds. New measures are suggested to provide stable LOAS by introducing controller for unstable states which bring positive eigenvalues in negative half plan hence stability of LOAS is guaranteed and error bound also exist. The suggested techniques gives better results than existing LOAS techniques for stability preservation.

2.1 Preliminaries

Consider a discrete time system be given as:

$$x(k+1) = \tilde{A}x(k) + \tilde{B}u(k) \tag{2.1}$$

$$y = \tilde{C}x(k) + \tilde{D}u(k)$$

$$\tilde{G}(z) = \tilde{C}(zI - \tilde{A})^{-1}\tilde{B} + D, \tag{2.2}$$

where $\tilde{A} \in R^{n\times n}$, $\tilde{B} \in R^{n\times m}$, $\tilde{C} \in R^{p\times n}$, $D \in R^{p\times m}$ and $\{\tilde{A}, \tilde{B}, \tilde{C}, D\}$ is its n^{th} order minimal realization with p outputs and m inputs. The LOAS

$$\begin{aligned} x_r(k+1) &= \tilde{A}_r x_r(k) + \tilde{B}_r u(k) \\ y_r(k) &= \tilde{C}_r x_r(k) + D_r u(k) \end{aligned} \tag{2.3}$$

$$\tilde{G}_r(z) = \tilde{C}_r(zI - \tilde{A}_r)^{-1}\tilde{B}_r + D_r, \tag{2.4}$$

is obtained by approximating the actual system (in the desired limited frequency range $[\omega_1, \omega_2]$) where $\omega_2 > \omega_1$, where $\{\tilde{A}_r \in R^{r\times r}, \tilde{B}_r \in R^{r\times m}, \tilde{C}_r \in R^{p\times r}, D_r \in R^{p\times m}\}$ with

$r < n$.

$$\tilde{P}_c = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{j\omega}I - \tilde{A})^{-1}\tilde{B}\tilde{B}^T(e^{-j\omega}I - \tilde{A}^T)^{-1}d\omega \quad (2.5)$$

$$\tilde{Q}_o = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{-j\omega}I - \tilde{A}^T)^{-1}\tilde{C}^T\tilde{C}(e^{j\omega}I - \tilde{A})^{-1}d\omega \quad (2.6)$$

$\tilde{P}_c$ and $\tilde{Q}_o$ satisfy

$$\tilde{A}\tilde{P}_c\tilde{A}^T - \tilde{P}_c + \tilde{B}\tilde{B}^T = 0 \quad (2.7)$$

$$\tilde{A}^T\tilde{Q}_o\tilde{A} - \tilde{Q}_o + \tilde{C}^T\tilde{C} = 0 \quad (2.8)$$

Using the Parseval's relationship

$$\tilde{P}_c(\omega) = \frac{1}{2\pi}\int_{-\omega}^{+\omega}(e^{j\omega}I - \tilde{A})^{-1}\tilde{B}\tilde{B}^T(e^{j\omega}I - \tilde{A}^T)^{-1}d\omega$$

$$\tilde{Q}_o(\omega) = \frac{1}{2\pi}\int_{-\omega}^{+\omega}(e^{-j\omega}I - \tilde{A}^T)^{-1}\tilde{C}^T\tilde{C}(e^{-j\omega}I - \tilde{A})^{-1}d\omega$$

2.1.1 WZ's Technique [41]

Let $\tilde{P}_W$ and $\tilde{Q}_W$ be defined as

$$\tilde{P}_W(\omega) = \frac{1}{2\pi}\int_{-\omega}^{+\omega}(e^{j\omega}I - \tilde{A})^{-1}\tilde{B}\tilde{B}^T(e^{j\omega}I - \tilde{A}^T)^{-1}d\omega$$

$$\tilde{Q}_W(\omega) = \frac{1}{2\pi}\int_{-\omega}^{+\omega}(e^{-j\omega}I - \tilde{A}^T)^{-1}\tilde{C}^T\tilde{C}(e^{-j\omega}I - \tilde{A})^{-1}d\omega$$

These Gramians P_W and Q_W satisfy

$$\tilde{A}\tilde{P}_W\tilde{A}^T - \tilde{P}_W + \tilde{X}_W = 0 \quad (2.9)$$

$$\tilde{A}^T\tilde{Q}_W\tilde{A} - \tilde{Q}_W + \tilde{Y}_W = 0 \quad (2.10)$$

where

$$\tilde{X}_W = \tilde{B}\tilde{B}^T F^* + F\tilde{B}\tilde{B}^T \quad (2.11)$$

$$\tilde{Y}_W = \tilde{C}^T\tilde{C}F + F^*\tilde{C}^T\tilde{C} \quad (2.12)$$

$$F = -\frac{\omega_2 - \omega_1}{4\pi}I + \frac{1}{2\pi}\int_{\delta\omega}(e^{j\omega}I - \tilde{A})^{-1}d\omega \quad (2.13)$$

F^* is conjugate transpose of F, $\delta\omega$ is the integration of interval $[\omega_1, \omega_2]$. Let

$$T_W^T\tilde{Q}_W T_W = T_W^{-1}\tilde{P}_W T_W^{-T} = diag(\sigma_1, \sigma_2, \cdots, \sigma_n) \quad (2.14)$$

where $\sigma_j \geq \sigma_{j+1},\ j = 1, 2, \ldots, n-1,\ \sigma_r > \sigma_{r+1}$ and T_W is used to get LOAS by transforming the actual system into a balanced realization. The LOAS obtained $\tilde{G}_r(z) = \tilde{C}_r(zI - \tilde{A}_r)^{-1}\tilde{B}_r + D_r$.

$$T_W^{-1}\tilde{A}T_W = \begin{bmatrix} A_r & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\quad T_W^{-1}\bar{B} = \begin{bmatrix} B_r \\ B_2 \end{bmatrix} \quad (2.15)$$

$$\tilde{C}T_W = \begin{bmatrix} C_r & C_2 \end{bmatrix},\ D = D_r \quad (2.16)$$

The possibly indefinite matrices X_W and Y_W can be decomposed as

$$\tilde{X}_W = US_WU^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix}\begin{bmatrix} S_{W_1} & 0 \\ 0 & S_{W_2} \end{bmatrix}\begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} \quad (2.17)$$

$$\tilde{Y}_W = VR_WV^T = \begin{bmatrix} V_1 & V_2 \end{bmatrix}\begin{bmatrix} R_{W_1} & 0 \\ 0 & R_{W_2} \end{bmatrix}\begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \quad (2.18)$$

where

$S_{W_1} = diag(s_1, \cdots, s_{l-1}) \geq 0$, $S_{W_2} = diag(s_l, \cdots, s_n) < 0$, $R_{W_1} = diag(r_1, \cdots, r_{p-1)} \geq$

0, $R_{W_2} = diag(r_p, \cdots, r_n) < 0$. $(l-1)$ and $(p-1)$ are the singular values which is positive of $\tilde{X}_W$ and $\tilde{Y}_W$ respectively.

Remark 1 *Since X_W and Y_W are negative/semi-negative definite, which results unstable LOAS [42].*

2.1.2 Existing Stability Preserving Frequency Limited Techniques

Let P_F and Q_F satisfy

$$\tilde{A} P_F \tilde{A}^T - P_F + B_F B_F^T = 0 \quad (2.19)$$

$$\tilde{A}^T Q_F \tilde{A} - Q_F + C_F^T C_F = 0 \quad (2.20)$$

$$B_{S_1} = U \begin{bmatrix} S_{W_1}^{1/2} & 0 \\ 0 & |S_{W_2}|^{1/2} \end{bmatrix}$$

$$B_{S_2} = U \begin{bmatrix} S_{W_1}^{1/2} & 0 \\ 0 & 0 \end{bmatrix}$$

$$B_I = \begin{cases} U(S_W - s_n I)^{1/2} & \text{for } s_n < 0 \\ U S_W^{1/2} & \text{for } s_n \geq 0 \end{cases}$$

$$C_{S_1} = \begin{bmatrix} R_{W_1}^{1/2} & 0 \\ 0 & |R_{W_2}|^{1/2} \end{bmatrix} V^T$$

$$C_{S_2} = \begin{bmatrix} R_{W_1}^{1/2} & 0 \\ 0 & 0 \end{bmatrix} V^T$$

$$C_I = \begin{cases} (R_W - r_n I)^{1/2} V^T & \text{for } r_n < 0 \\ R_W^{1/2} V^T & \text{for } r_n \geq 0. \end{cases}$$

Let T_F is obtained as:

$$T_F^T Q_F T_F = T_F^{-1} P_F T_F^{-T} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}$$

The LOAS are obtained after transforming the original system using the matric T_F similar to equations $(13) - (14)$. These LOAS which are obtained here are stable guaranteed and their error bounds also exists. Let new virtual/fictitious controllability $\tilde{P}_{MI}$ and observability $\tilde{Q}_{MI}$ Gramians are computed as

$$\tilde{A} P_{MI} \tilde{A}^T - \tilde{P}_{MI} + B_{MI} B_{MI}^T = 0 \quad (2.21)$$

$$\tilde{A}^T Q_{MI} \tilde{A} - \tilde{Q}_{MI} + C_{MI}^T C_{MI} = 0 \quad (2.22)$$

The new virtual matrices B_{MI} and C_{MI} where

$$B_{MI} = U \begin{bmatrix} S_{W_1}^{1/2} & 0 \\ 0 & S_{MI_2}^{1/2} \end{bmatrix} = US_{MI}^{1/2},$$

$$C_{MI} = \begin{bmatrix} R_{W_1}^{1/2} & 0 \\ 0 & R_{MI_2}^{1/2} \end{bmatrix} V^T = R_{MI}^{1/2} V^T$$

where

$$S_{MI_2} = \begin{cases} (S_{W_2}.s_l) & \text{for } l < n \end{cases}$$

$$R_{MI_2} = \begin{cases} (R_{W_2}.r_p) & \text{for } p < n \end{cases}$$

Let similarity transformation matrix T_{MI} is calculated as

$$T_{MI}^T Q_{MI} T_{MI} = T_{MI}^{-1} P_{MI} T_{MI}^{-T} = diag\{\sigma_1, \sigma_2, \sigma_3 \ldots, \sigma_n\}$$

where $\sigma_j \geq \sigma_{j+1}$ and $\sigma_r \geq \sigma_{r+1}$. The LOAS is obtained as

$$T_{MI}^{-1} \tilde{A} T_{MI} = \begin{bmatrix} A_r & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad T_{MI}^{-1} \tilde{B} = \begin{bmatrix} B_r \\ B_2 \end{bmatrix} \tag{2.23}$$

$$\tilde{C} T_{MI} = \begin{bmatrix} C_r & C_2 \end{bmatrix}, \ D = D_r \tag{2.24}$$

Let an contragredient transformation matrix T (as used in transforming original native system) is obtained as

$$T^T Q_{MI} T = T^{-1} P_{MI} T^{-T} = \begin{bmatrix} \varsigma_1 & 0 & \cdots & 0 \\ 0 & \varsigma_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \varsigma_n \end{bmatrix}$$

where $\varsigma_j \geq \varsigma_{j+1},\ j = 1, 2, 3, \ldots, n-1$, $\varsigma_k > \varsigma_{k+1}$. Lower order approximated systems are calculated by applying transformation matrix and balancing the transformed realization.

Remark 2 *Since $\tilde{X}_W \leq B_{MI} B_{MI}^T \geq 0$, $\tilde{Y}_W \leq C_{MI}^T C_{MI} \geq 0$, $P_{MI} > 0$ and $Q_{MI} > 0$. Which implies the minimality of the realization $(\tilde{A}, B_{MI}, C_{MI})$ and stability of the LOAS is guaranteed.*

Theorem 1 *Let $rank \begin{bmatrix} B_{MI} & \tilde{B} \end{bmatrix} = rank\,[B_{MI}]$ and $rank \begin{bmatrix} C_{MI} \\ \tilde{C} \end{bmatrix} = rank\,[C_{MI}]$ following error bound holds*

$$\|\tilde{G}(z) - \tilde{G}_r(z)\|_\infty \leq 2\|L_{MI}\|\|K_{MI}\| \sum_{j=r+1}^{n} \sigma_j$$

where

$$L_{MI} = \begin{cases} CVR_{MI}^{-1/2} & \textit{if } R_{W_2} \textit{ exists} \\ CVR_W^{-1/2} & \textit{otherwise} \end{cases}$$

$$K_{MI} = \begin{cases} S_{MI}^{-1/2}U^T B & \textit{if } S_{W_2} \textit{ exists} \\ S_W^{-1/2}U^T B & \textit{otherwise} \end{cases}$$

Proof: Since $rank\begin{bmatrix}B_{MI} & \tilde{B}\end{bmatrix} = rank\,[B_{MI}]$ and $rank\begin{bmatrix}C_{MI} \\ \tilde{C}\end{bmatrix} = rank\,[C_{MI}]$, the relationships $B = B_{MI}K_{MI}$ and $C = L_{MI}C_{MI}$ hold. By partitioning $B_{MI} = \begin{bmatrix}B_{MI_1} \\ B_{MI_2}\end{bmatrix}$, $C_{MI} = \begin{bmatrix}C_{MI_1} & C_{MI_2}\end{bmatrix}$ and substituting $B_r = B_{MI_1}K_{MI}$, $C_r = L_{MI}C_{MI_1}$ respectively yields

$$\begin{aligned}
\|\tilde{G}(z) - \tilde{G}_r(z)\|_\infty &= \|\tilde{C}(zI-\tilde{A})^{-1}\tilde{B}-\tilde{C}_r(zI-\tilde{A}_r)^{-1}\tilde{B}_r\|_\infty \\
&= \|L_{MI}C_{MI}(zI-\tilde{A})^{-1}B_{MI}K_{MI} \\
&\quad -L_{MI}C_{MI_1}(zI-\tilde{A}_r)^{-1}B_{MI_1}K_{MI}\|_\infty \\
&= \|L_{MI}(C_{MI}(zI-\tilde{A})^{-1}B_{MI} \\
&\quad -C_{MI_1}(zI-\tilde{A}_r)^{-1}B_{MI_1})K_{MI}\|_\infty \\
&\leq \|L_{MI}\|\|(C_{MI}(zI-\tilde{A})^{-1}B_{MI} \\
&\quad -C_{MI_1}(zI-\tilde{A}_r)^{-1}B_{MI_1})\|_\infty\|K_{MI}\|
\end{aligned}$$

If $\{\tilde{A}_r, B_{MI_1}, C_{MI_1}\}$ is LOAS obtained from original system $\{\tilde{A}, B_{MI}, C_{MI}\}$.

$$\|(C_{MI}(zI-\tilde{A})^{-1}B_{MI}-C_{MI_1}(zI-\tilde{A}_r)^{-1}B_{MI_1})\|_\infty \leq 2\sum_{j=r+1}^{n}\sigma_j.$$

Therefore,

$$\|\tilde{G}(z) - \tilde{G}_r(z)\|_\infty \leq 2\|L_{MI}\|\|K_{MI}\| \sum_{j=r+1}^{n} \sigma_j$$

Remark 3 *$X_G \geq 0$ and $Y_G \geq 0$ which implies $P_G = P_F = P_{MI}$ and $Q_G = Q_F = Q_{MI}$. Otherwise $P_G < P_{MI}$ and $Q_G < Q_{MI}$. Moreover, Hankel values satisfy* : $(\lambda_j[P_G Q_G])^{1/2} \leq (\lambda_j[P_{MI} Q_{MI}])^{1/2}$

2.2 Main Results

The LOAS are obtained after transforming the original system using the matric T_F similar to equations $(13) - (14)$. These LOAS have their stability guaranteed and have error bounds. The existing stability preserving techniques [42], [40] modified X_G and Y_G to ensure positive/semipositive definite.

Let controllability P_{EI} and observability Q_{EI} Gramians are computed as

$$\tilde{A} P_{EI} \tilde{A}^T - P_{EI} + B_{EI} B_{EI}^T = 0 \quad (2.25)$$

$$\tilde{A}^T Q_{EI} \tilde{A} - Q_{EI} + C_{EI}^T C_{EI} = 0 \quad (2.26)$$

The input and output related matrices are given as B_{EI} and C_{EI}. Let similarity transformation matrix T_{EI} is calculated as

$$T_{EI}^T Q_{EI} T_{EI} = T_{EI}^{-1} P_{EI} T_{EI}^{-T} = diag\{\sigma_1, \sigma_2, \sigma_3 \ldots, \sigma_n\}$$

where $\sigma_j \geq \sigma_{j+1}$ and $\sigma_r \geq \sigma_{r+1}$. The LOAS is obtained as

$$T_{EI}^{-1}\tilde{A}T_{EI} = \begin{bmatrix} A_r & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad T_{EI}^{-1}\tilde{B} = \begin{bmatrix} B_r \\ B_2 \end{bmatrix} \tag{2.27}$$

$$\tilde{C}T_{EI} = \begin{bmatrix} C_r & C_2 \end{bmatrix}, \; D = D_r \tag{2.28}$$

T_{EI} (used to transform the original system) is obtained as

$$T_{EI}^{T}Q_{EI}T_{EI} = T_{EI}^{-1}P_{EI}T_{EI}^{-T} = \begin{bmatrix} \varsigma_1 & 0 & \cdots & 0 \\ 0 & \varsigma_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \varsigma_n \end{bmatrix}$$

where $\varsigma_j \geq \varsigma_{j+1},\; j = 1, 2, 3, \ldots, n-1,\; \varsigma_k > \varsigma_{k+1}$. Lower order approximated models are calculated by applying transformation matrix and partitioning the transformed realization.

Remark 4 *Since $\tilde{X}_W = B_{EI}B_{EI}^T \leq 0$, $\tilde{Y}_W = C_{EI}^T C_{EI} \leq 0$, $P_{EI} \leq 0$ and $Q_{EI} \leq 0$. Therefore, the realization is minimal $(\tilde{A}, B_{EI}, C_{EI})$ is not stable and stability of lower order approximated system is not ensured.*

Design of Feedback Controller for Lower Order Approximated System:

Let LOAS,

$$\begin{aligned} x_r(k+1) &= \tilde{A}_r x_r(k) + \tilde{B}_r u(k) \\ y_r(k) &= \tilde{C}_r x_r(k) + D_r u(k) \end{aligned} \tag{2.29}$$

$$\tilde{G}_r(z) = \tilde{C}_r(zI - \tilde{A}_r)^{-1}\tilde{B}_r + D_r, \tag{2.30}$$

is obtained by approximating the actual system (in the desired limited frequency range $[\omega_1, \omega_2]$) where $\omega_2 > \omega_1$, where $\{\tilde{A}_r \in R^{r\times r}, \tilde{B}_r \in R^{r\times m}, \tilde{C}_r \in R^{p\times r}, D_r \in R^{p\times m}\}$ with $r < n$. To make LOAS stable introducing State Feedback Controller and Output Feedback Controller with states Feedback using unstable lower order approximated system as given above, lower order control law is applying in above system $u(k) = -K_r x_r + \hat{u}$ to feedback its lower order states to stabilizing the lower order approximated system which is given following,

$$\begin{aligned} x_r(k+1) &= \tilde{A}_r x_r(k) + \tilde{B}_r(-K_r x_r + \hat{u}) \\ y_r(k) &= \tilde{C}_r x_r(k) + D_r(-K_r x_r + \hat{u}) \end{aligned}$$

$$\begin{aligned} x_r(k+1) &= (\tilde{A}_r - \tilde{B}_r K_r)x_r + \tilde{B}_r\hat{u} \\ y_r(k) &= (\tilde{C}_r - D_r K_r)x_r + D_r\hat{u} \end{aligned}$$

$$\hat{G}_r(z) = (\tilde{C}_r - D_r K_r)(zI - (\tilde{A}_r - \tilde{B}_r K_r))^{-1}\tilde{B}_r + D_r,$$

where $\{(\tilde{A}_r - \tilde{B}_r K_r) = \hat{A}_r \in R^{r\times r}, \tilde{B}_r = \hat{B}_r \in R^{r\times m}, (\tilde{C}_r - D_r K_r) = \hat{C}_r \in R^{p\times r}, D_r = \hat{D}_r \in R^{p\times m}\}$ with $r < n$. Stable lower order approximated system will be given as,

$$\begin{aligned} \hat{x_r}(k+1) &= \hat{A}_r x_r + \hat{B}_r\hat{u} \\ \hat{y_r}(k) &= \hat{C}_r x_r + \hat{D}_r\hat{u} \end{aligned} \tag{2.31}$$

$$\hat{G}_r(z) = \hat{C}_r(zI - \hat{A}_r)^{-1}\hat{B}_r + \hat{D}_r, \tag{2.32}$$

Lower Order Approximated System is also give in Fig. 2.1

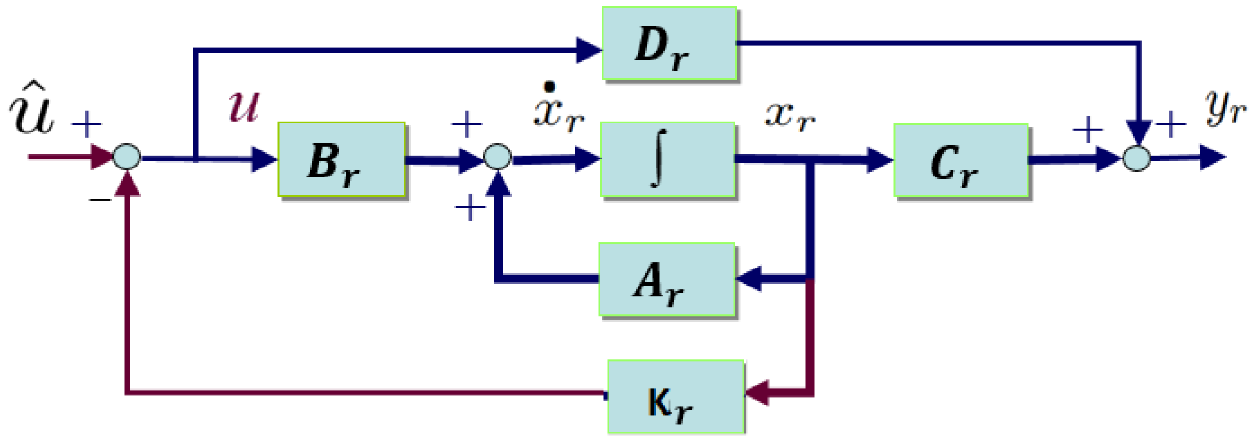

Figure 2.1: State Feedback Control for Lower Order Approximated System

Remark 5 *Since* $\tilde{X}_W = B_{EI}B_{EI}^T \leq 0$, $\tilde{Y}_W = C_{EI}^T C_{EI} \leq 0$, $P_{EI} \leq 0$ *and* $Q_{EI} \leq 0$. *Which implies the minimality of the realization* $(\tilde{A}, B_{EI}, C_{EI})$ *and stability of* $(\hat{A}_r, \hat{B}_r, \hat{C}_r)$ *lower order approximated system is guaranteed given in Fig. 2.1.*

Design of Observer Based Feedback Controller for Lower Order Approximated System:

Let LOAS,

$$\begin{aligned} x_r(k+1) &= \tilde{A}_r x_r(k) + \tilde{B}_r u(k) \\ y_r(k) &= \tilde{C}_r x_r(k) + D_r u(k) \end{aligned} \tag{2.33}$$

$$\tilde{G}_r(z) = \tilde{C}_r(zI - \tilde{A}_r)^{-1}\tilde{B}_r + D_r, \tag{2.34}$$

is obtained by approximating the actual system (in the desired limited frequency range $[\omega_1, \omega_2]$) where $\omega_2 > \omega_1$, where $\{\tilde{A}_r \in R^{r\times r}, \tilde{B}_r \in R^{r\times m}, \tilde{C}_r \in R^{p\times r}, D_r \in R^{p\times m}\}$ with

$r < n$.

To make LOAS stable introducing Observer Based State Feedback Controller using unstable lower order approximated system as given above,

Lower Order Approximated Observer:

$$\begin{aligned}\hat{x_r}(k+1) &= \tilde{A}_r\hat{x_r}(k) + L(y_r - \tilde{C}_r\hat{x_r}) + \tilde{B}_r u(k)\\ \hat{y_r}(k) &= \tilde{C}_r\hat{x_r}(k) + D_r u(k)\end{aligned}$$

State Feedback Law:

$u(k) = -K_r x_r + \hat{u}$

lower order control law is applying in above system $u(k) = -K_r x_r + \hat{u}$ to feedback its observer based lower order states to stabilizing the lower order approximated system which is given following,

$$\begin{aligned}\hat{x_r}(k+1) &= (\tilde{A}_r - LC_r)\hat{x_r}(k) + L\tilde{C}_r x_r + \tilde{B}_r(-K_r\hat{x_r} + \hat{u})\\ \hat{y_r}(k) &= \tilde{C}_r\hat{x_r}(k) + D_r(-K_r\hat{x_r} + \hat{u})\end{aligned}$$

$$\begin{aligned}\hat{x_r}(k+1) &= (\tilde{A}_r - L\tilde{C}_r - \tilde{B}_r K_r)\hat{x_r} + \tilde{B}_r\hat{u}\\ \hat{y_r}(k) &= (\tilde{C}_r - D_r K_r)\hat{x_r} + D_r\hat{u}\end{aligned}$$

$$\hat{G_{ro}}(z) = (\tilde{C}_r - D_r K_r)(zI - (\tilde{A}_r - L\tilde{C}_r - \tilde{B}_r K_r))^{-1}\tilde{B}_r + D_r,$$

where $\{(\tilde{A}_r - L\tilde{C}_r - \tilde{B}_r K_r) = \hat{A_{ro}} \in R^{r\times r}, \tilde{B}_r = \hat{B_{ro}} \in R^{r\times m}, (\tilde{C}_r - D_r K_r) = \hat{C_{ro}} \in R^{p\times r}, D_r = \hat{D_r o} \in R^{p\times m}\}$ with $r < n$. Stable lower order approximated system will be given as,

$$\begin{aligned} \hat{x_r}(k+1) &= \hat{A_{ro}} x_r + \hat{B_{ro}} \hat{u} \\ \hat{y_r}(k) &= \hat{C_{ro}} x_r + \hat{D_r o} \hat{u} \end{aligned} \tag{2.35}$$

$$\hat{G_{ro}}(z) = \hat{C_{ro}}(zI - \hat{A_{ro}})^{-1}\hat{B_{ro}} + \hat{D_{ro}}, \tag{2.36}$$

Observer Based Lower Order Approximated System is also give in Fig. 2.2

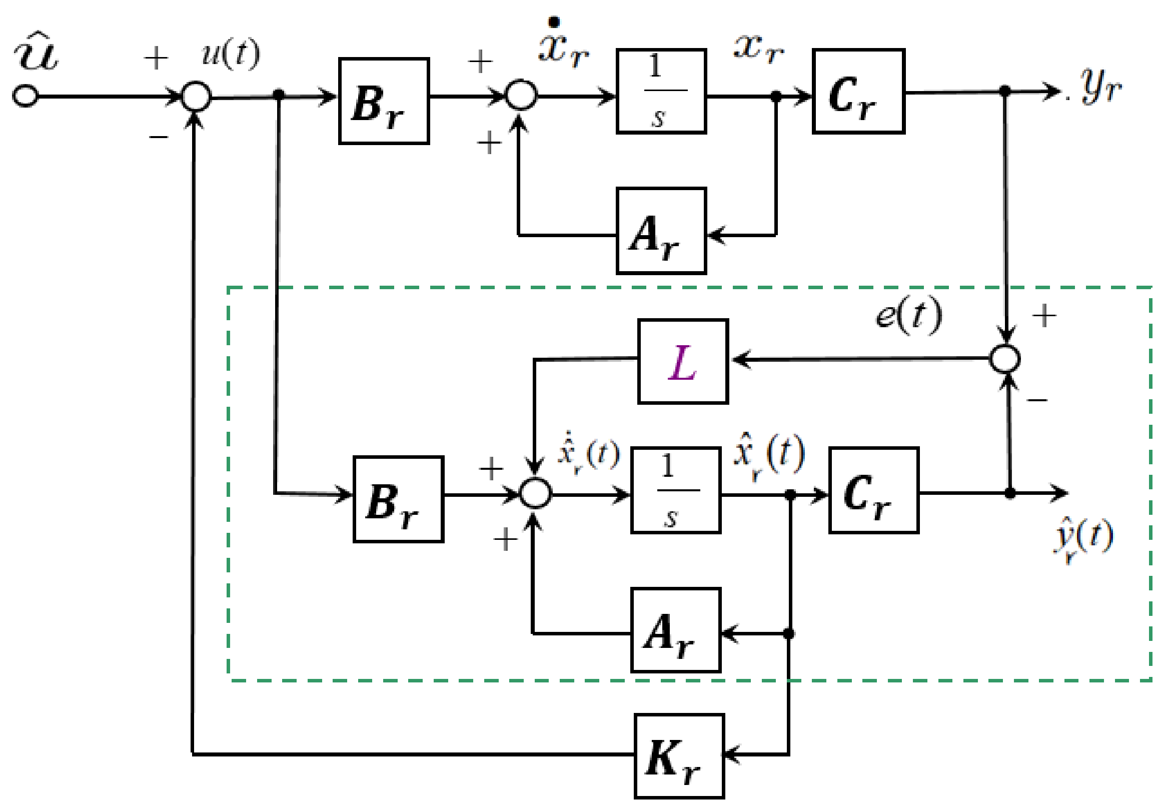

Figure 2.2: Observer Based State Feedback Control for Lower Order Approximated System

Remark 6 *Since* $\tilde{X}_W = B_{EI}B_{EI}^T \leq 0$, $\tilde{Y}_W = C_{EI}^T C_{EI} \leq 0$, $P_{EI} \leq 0$ *and* $Q_{EI} \leq 0$. *Which results the minimality of the realization* $(\tilde{A}, B_{EI}, C_{EI})$ *and stability of* $(\hat{A_{ro}}, \hat{B_{ro}}, \hat{C_{ro}})$ *lower order approximated system is guaranteed given in Fig. 2.2.*

Theorem 2 *Let* $rank \begin{bmatrix} B_{EI} & \tilde{B} \end{bmatrix} = rank \left[B_{EI} \right]$ *and* $rank \begin{bmatrix} C_{EI} \\ \tilde{C} \end{bmatrix} = rank \left[C_{EI} \right]$ *following error bound holds*

$$\|\tilde{G}(z) - \hat{G}_r(z)\|_\infty \leq 2\|L_{EI}\|\|K_{EI}\| \sum_{j=r+1}^{n} \sigma_j$$

where

$$L_{EI} = \left\{ \; CVR_W^{-1/2} \right.$$

$$K_{EI} = \left\{ \; S_W^{-1/2}U^T B \right.$$

Proof: Since $rank \begin{bmatrix} B_{EI} & \tilde{B} \end{bmatrix} = rank \left[B_{EI} \right]$ and $rank \begin{bmatrix} C_{EI} \\ \tilde{C} \end{bmatrix} = rank \left[C_{EI} \right]$, the relationships $B = B_{EI}K_{EI}$ and $C = L_{EI}C_{EI}$ hold. By partitioning $B_{EI} = \begin{bmatrix} B_{EI_1} \\ B_{EI_2} \end{bmatrix}$, $C_{EI} =$

$\begin{bmatrix} C_{EI_1} & C_{EI_2} \end{bmatrix}$ and substituting $\hat{B}_r = B_{EI_1}K_{EI}$, $\hat{C}_r = L_{EI}C_{EI_1}$ respectively yields

$$\begin{aligned}
\|\tilde{G}(z) - \hat{G}_r(z)\|_\infty &= \|\tilde{C}(zI-\tilde{A})^{-1}\tilde{B}-\hat{C}_r(zI-\hat{A}_r)^{-1}\hat{B}_r\|_\infty \\
&= \|L_{EI}C_{EI}(zI-\tilde{A})^{-1}B_{EI}K_{EI} \\
&\quad -L_{EI}C_{EI_1}(zI-\hat{A}_r)^{-1}B_{EI_1}K_{EI}\|_\infty \\
&= \|L_{EI}(C_{EI}(zI-\tilde{A})^{-1}B_{EI} \\
&\quad -C_{EI_1}(zI-\hat{A}_r)^{-1}B_{EI_1})K_{EI}\|_\infty \\
&\leq \|L_{EI}\|\|(C_{EI}(zI-\tilde{A})^{-1}B_{EI} \\
&\quad -C_{EI_1}(zI-\hat{A}_r)^{-1}B_{EI_1})\|_\infty\|K_{EI}\|
\end{aligned}$$

If $\{\hat{A}_r, B_{EI_1}, C_{EI_1}\}$ is LOAS obtained from original system $\{\tilde{A}, B_{EI}, C_{EI}\}$.

$$\|(C_{EI}(zI-\tilde{A})^{-1}B_{EI} - C_{EI_1}(zI-\hat{A}_r)^{-1}B_{EI_1})\|_\infty \leq 2\sum_{j=r+1}^{n}\sigma_j.$$

Therefore,

$$\|\tilde{G}(z) - \hat{G}_r(z)\|_\infty \leq 2\|L_{EI}\|\|K_{EI}\| \sum_{j=r+1}^{n} \sigma_j$$

Theorem 3 *Let* $rank\begin{bmatrix} B_{EI} & \tilde{B} \end{bmatrix} = rank\,[B_{EI}]$ *and* $rank\begin{bmatrix} C_{EI} \\ \tilde{C} \end{bmatrix} = rank\,[C_{EI}]$ *following error bound holds*

$$\|\tilde{G}(z) - \hat{G}_r(z)\|_\infty \leq 2\|L_{EI}\|\|K_{EI}\| \sum_{j=r+1}^{n} \sigma_j$$

where

$$L_{EI} = \left\{ \; CVR_W^{-1/2} \right.$$

$$K_{EI} = \left\{ \; S_W^{-1/2} U^T B \right.$$

Proof: Since $rank \begin{bmatrix} B_{EI} & \tilde{B} \end{bmatrix} = rank\,[B_{EI}]$ and $rank \begin{bmatrix} C_{EI} \\ \tilde{C} \end{bmatrix} = rank\,[C_{EI}]$, the relationships $B = B_{EI}K_{EI}$ and $C = L_{EI}C_{EI}$ hold. By partitioning $B_{EI} = \begin{bmatrix} B_{EI_1} \\ B_{EI_2} \end{bmatrix}, C_{EI} = \begin{bmatrix} C_{EI_1} & C_{EI_2} \end{bmatrix}$ and substituting $\hat{B_r o} = B_{EI_1}K_{EI}$, $\hat{C_r o} = L_{EI}C_{EI_1}$ respectively yields

$$\begin{aligned}
\|\tilde{G}(z) - \hat{G_r o}(z)\|_\infty &= \|\tilde{C}(zI-\tilde{A})^{-1}\tilde{B}-\hat{C_r o}(zI-\hat{A_{ro}})^{-1}\hat{B_r o}\|_\infty \\
&= \|L_{EI}C_{EI}(zI-\tilde{A})^{-1}B_{EI}K_{EI} \\
&\quad -L_{EI}C_{EI_1}(zI-\hat{A_{ro}})^{-1}B_{EI_1}K_{EI}\|_\infty \\
&= \|L_{EI}(C_{EI}(zI-\tilde{A})^{-1}B_{EI} \\
&\quad -C_{EI_1}(zI-\hat{A_{ro}})^{-1}B_{EI_1})K_{EI}\|_\infty \\
&\leq \|L_{EI}\|\|(C_{EI}(zI-\tilde{A})^{-1}B_{EI} \\
&\quad -C_{EI_1}(zI-\hat{A_{ro}})^{-1}B_{EI_1})\|_\infty\|K_{EI}\|
\end{aligned}$$

If $\{\hat{A_{ro}}, B_{EI_1}, C_{EI_1}\}$ is LOAS obtained from original system $\{\tilde{A}, B_{EI}, C_{EI}\}$.

$$\|(C_{EI}(zI-\tilde{A})^{-1}B_{EI} - C_{EI_1}(zI-\hat{A_{ro}})^{-1}B_{EI_1})\|_\infty \leq 2\sum_{j=r+1}^{n}\sigma_j.$$

Therefore,

$$\|\tilde{G}(z) - \hat{G_r o}(z)\|_\infty \leq 2\|L_{EI}\|\|K_{EI}\| \sum_{j=r+1}^{n} \sigma_j$$

Remark 7 *$X_G \geq 0$ and $Y_G \geq 0$ then $P_G = P_F = P_{EI}$ and $Q_G = Q_F = Q_{EI}$. Whereas, Hankel values satisfy* : $(\lambda_j[P_G Q_G])^{1/2} = (\lambda_j[P_{EI} Q_{EI}])^{1/2}$

2.3 Numerical Simulations

Numerical examples show the proposed technique effectiveness.

Example 1: Consider a transfer function of 6^{th} order Elliptic filter with band-pass $0.2001\pi - 0.35001\pi$, ripple is 0.2001 dB in the pass band and attenuation is 30 dB on the stop band.

$$G(z) = \frac{0.04278z^6 - 0.08375z^5 + 0.053z^4 + 9.499e^{-18}z^3 - 0.053z^2 + 0.08375z - 0.04278}{z^6 - 3.419z^5 + 6.098z^4 - 6.65z^3 + 4.761z^2 - 2.07z + 0.4722}$$

with desired frequency interval $0.03\pi - 0.09\pi$.

Analysis & Discussion

Fig. 2.3, 2.11 and 2.19 show the error $\sigma[G(s) - G_r(s)]$ plot, where $G_r(s)$ is 2^{nd}, 3^{rd} and 5^{th} LOAS obtained using existing and proposed techniques respectively. Fig. 2.4, 2.12 and 2.20 represent zoom-in view of frequency response error. Whereas, Fig. (2.5, 2.13, 2.21), (2.6, 2.14, 2.22), (2.7, 2.15, 2.23), (2.8, 2.16, 2.24), (2.9, 2.17, 2.25) and (2.10, 2.18, 2.26) represents impulse response, natural response, the step response, bode Plot, root locus plot and nyquist plot for the 2^{nd}, 3^{rd} and 5^{th} LOAS obtained using existing and proposed tech-

Table 2.1: LOAS for 2^{nd} order

Techniques	LOAS
Wang and Zilouchian	$\dfrac{0.04278z^2 - 0.09992z + 0.05708}{z^2 - 1.844z + 0.7537}$
Victor Sreeram	$\dfrac{0.04278z^2 - 0.04446z - 0.05361}{z^2 - 1.436z + 0.9041}$
Balance Truncation	$\dfrac{0.04278z^2 + 0.05752z - 0.1289}{z^2 - 1.241z + 0.8579}$
State Feedback Controller	$\dfrac{0.04278z^2 - 0.0597z + 0.03248}{z^2 - 0.9035z + 0.1785}$
Observer Based Controller	$\dfrac{0.04278z^2 - 0.05346z + 0.02866}{z^2 - 0.7577z + 0.08924}$

Table 2.2: LOAS for 3^{rd} order

Techniques	LOAS
Wang and Zilouchian	$\dfrac{0.04278z^3 - 0.143z^2 + 0.1583z - 0.05815}{z^3 - 2.854z^2 + 2.628z - 0.7687}$
Victor Sreeram	$\dfrac{0.04278z^3 + 0.01488z^2 - 0.1753z + 0.1178}{z^3 - 1.929z^2 + 1.628z - 0.4832}$
Balance Truncation	$\dfrac{0.04278z^3 + 0.04466z^2 - 0.2025z + 0.123}{z^3 - 1.957z^2 + 1.734z - 0.5957}$
State Feedback Controller	$\dfrac{0.04278z^3 - 0.07493z^2 + 0.06751z - 0.02804}{z^3 - 1.264z^2 + 0.5051z - 0.06498}$
Observer Based Controller	$\dfrac{0.04278z^3 - 0.06099z^2 + 0.05557z - 0.02596}{z^3 - 0.9378z^2 + 0.226z - 0.01624}$

Table 2.3: LOAS for 5^{th} order

Techniques	LOAS
Wang and Zilouchian	$\frac{0.04278z^5 - 0.09989z^4 - 0.05579z^3 + 0.3482z^2 - 0.3526z + 0.1173}{z^5 - 4.238z^4 + 7.549z^3 - 7.24z^2 + 3.744z - 0.879}$
Victor Sreeram	$\frac{0.04278z^5 - 0.09298z^4 + 0.1294z^3 - 0.2209z^2 + 0.2355z - 0.09125}{z^5 - 3.031z^4 + 4.518z^3 - 3.838z^2 + 1.895z - 0.4149}$
Balance Truncation	$\frac{0.04278z^5 - 0.08901z^4 + 0.1101z^3 - 0.1719z^2 + 0.1718z - 0.05649}{z^5 - 3.026z^4 + 4.569z^3 - 3.969z^2 + 2.022z - 0.4676}$
State Feedback Controller	$\frac{0.04278z^5 - 0.04909z^4 - 0.1986z^3 + 0.5279z^2 - 0.4639z + 0.1486}{z^5 - 3.051z^4 + 4.212z^3 - 3.04z^2 + 1.142z - 0.1478}$
Observer Based Controller	$\frac{0.04278z^5 - 0.04396z^4 - 0.213z^3 + 0.546z^2 - 0.4751z + 0.1517}{z^5 - 2.931z^4 + 3.874z^3 - 2.616z^2 + 0.8787z - 0.07388}$

Table 2.4: Poles location of the LOAS

Techniques	2^{nd} Order	3^{rd} Order	5^{th} Order
Wang and Zilouchian	$1.2319, 0.61181$	$1.1461, 1.0962, 0.61181$	$1.4274, 0.78826 \pm 0.553i, 0.61713 \pm 0.5323i$
Victor Sreeram	$0.72157 \pm 0.62803i$	$0.56229, 0.68317 \pm 0.62657i$	$0.5921, 0.75961 \pm 0.55445i, 0.45991 \pm 0.76215i$
Balance Truncation	$0.62046 \pm 0.68772i$	$0.69517, 0.6308 \pm 0.67747i$	$0.64938, 0.73835 \pm 0.57062i, 0.44979 \pm 0.7903i$
State Feedback Controller	$0.61181, 0.29172$	$0.31787, 0.33412, 0.61181$	$0.23994, 0.61713 \pm 0.5323i, 0.78826 \pm 0.553i$
Observer Based Controller	$0.61181, 0.14586$	$0.15894, 0.16706, 0.61181$	$0.11997, 0.61713 \pm 0.5323i, 0.78826 \pm 0.553i$

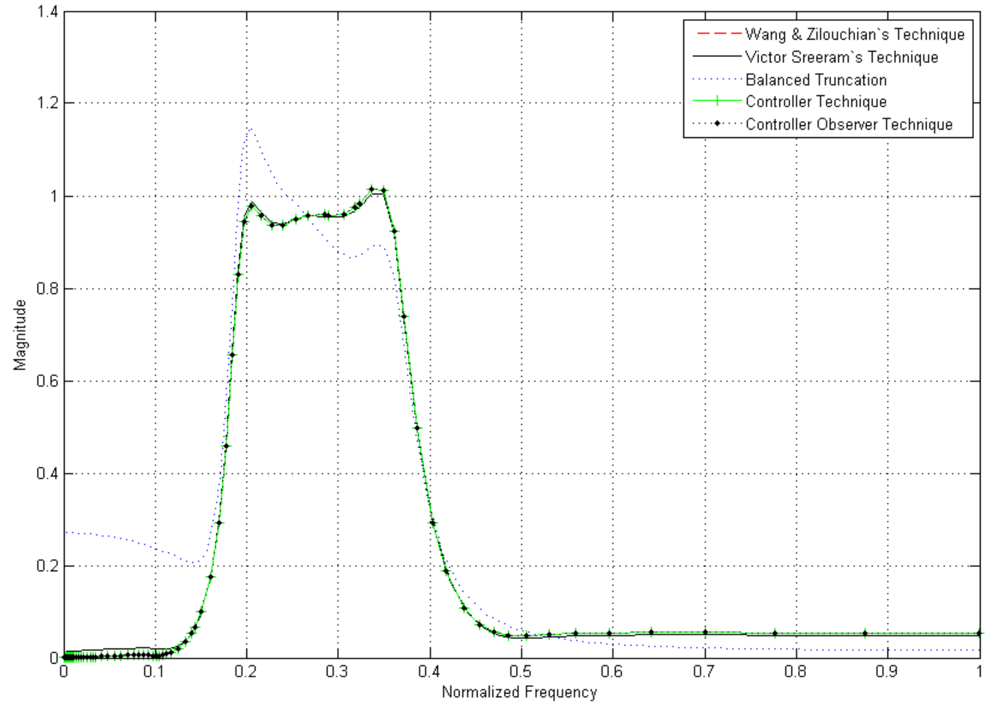

Figure 2.3: Error comparison for 2^{nd} order LOAS

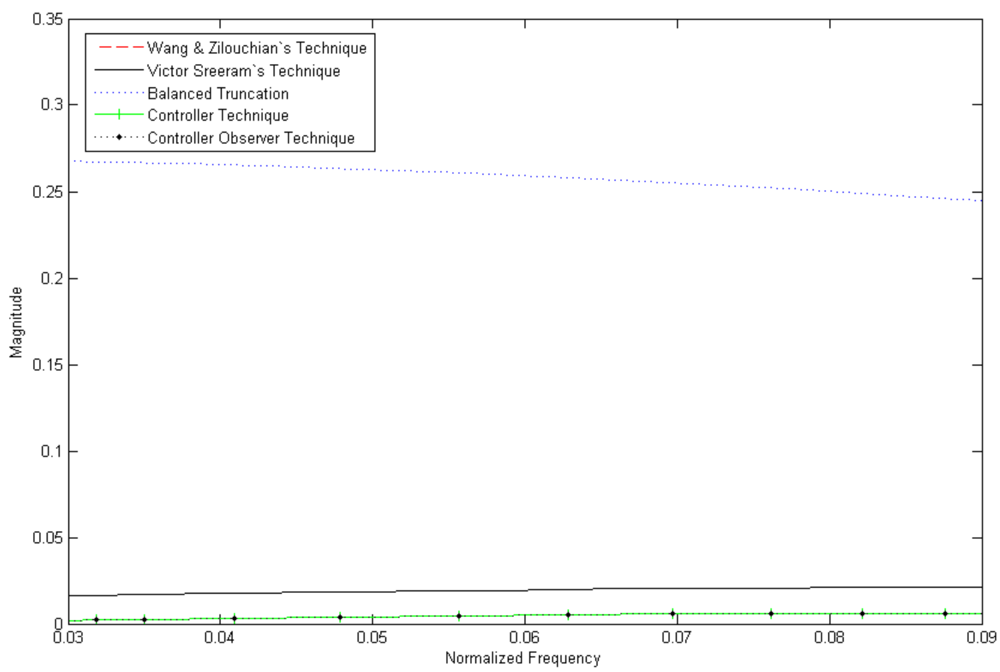

Figure 2.4: Error comparison - zoom-in view for 2^{nd} order LOAS

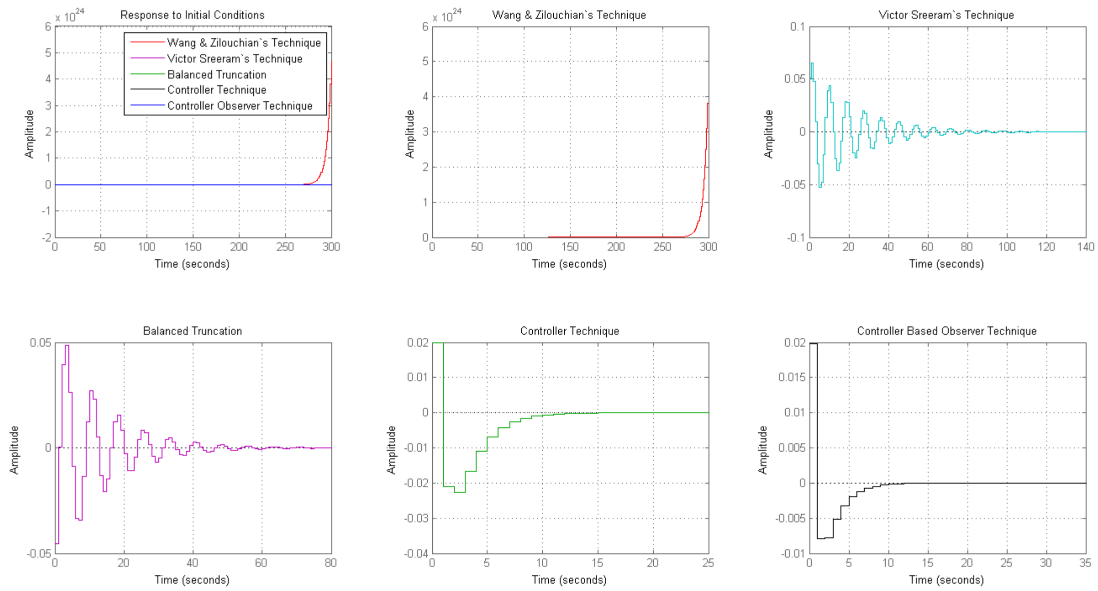

Figure 2.5: Natural response for 2^{nd} order LOAS

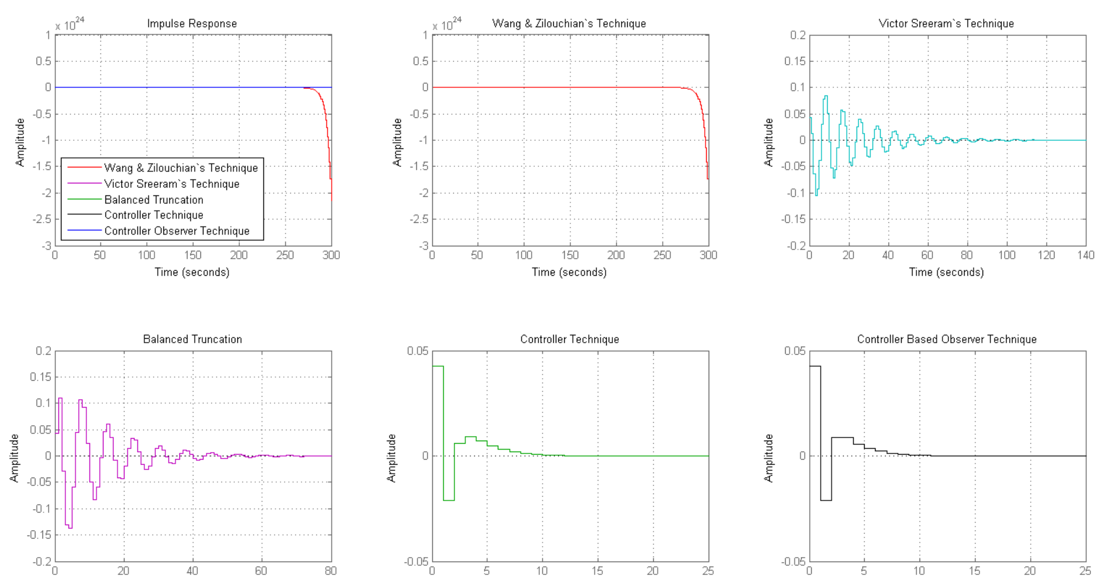

Figure 2.6: Impulse response for 2^{nd} order LOAS

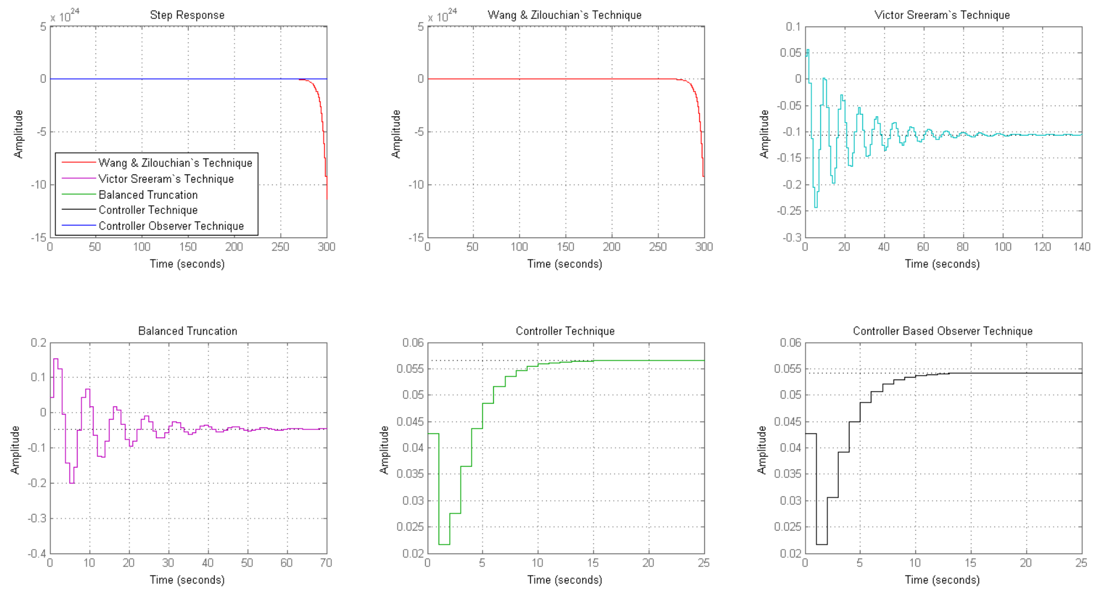

Figure 2.7: Step response for 2^{nd} order LOAS

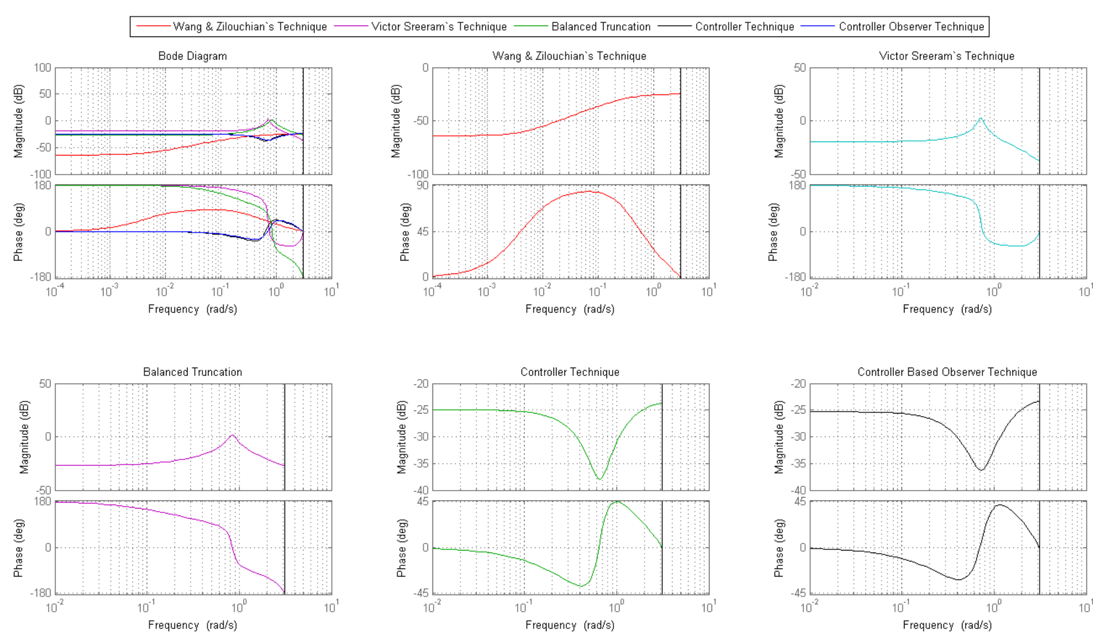

Figure 2.8: Bode plot for 2^{nd} order LOAS

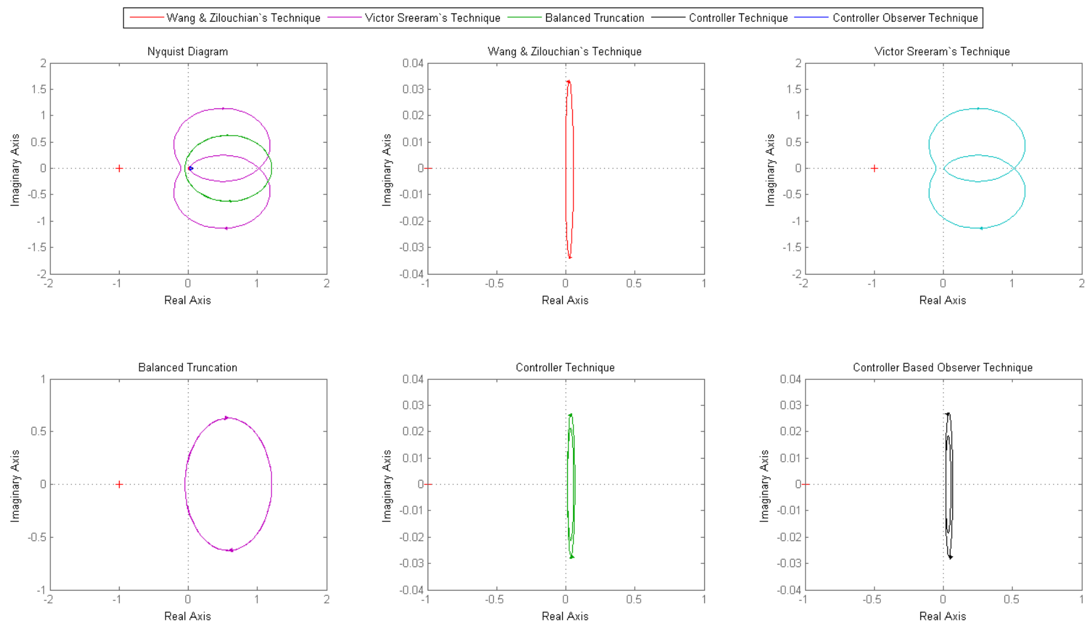

Figure 2.9: Nyquist plot for 2^{nd} order LOAS

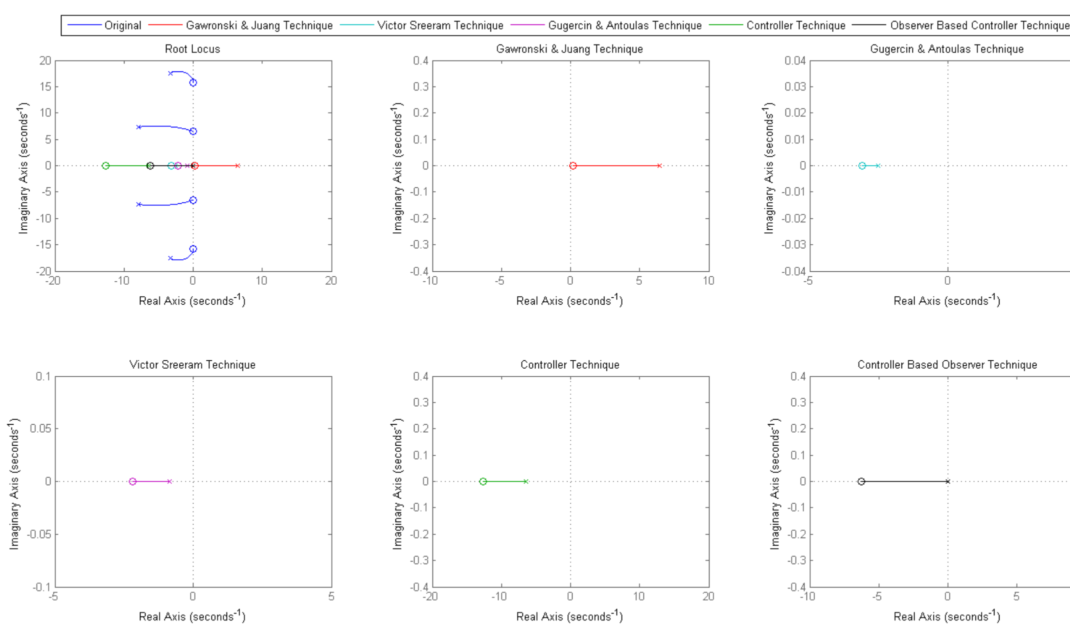

Figure 2.10: Root Locus plot for 2^{nd} order LOAS

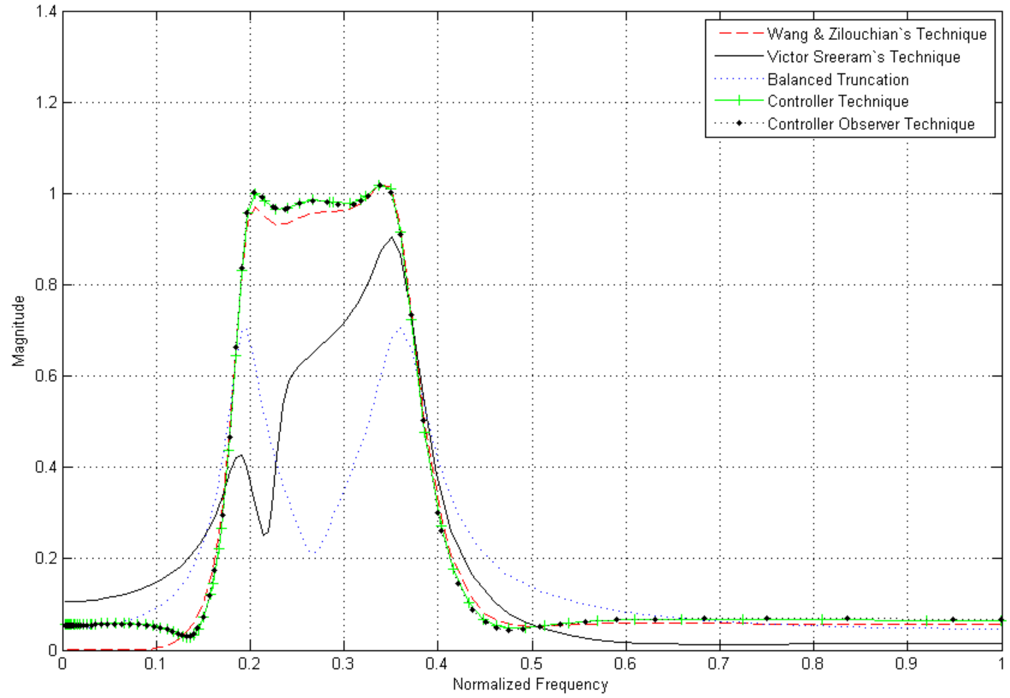

Figure 2.11: Error comparison for 3^{rd} order LOAS

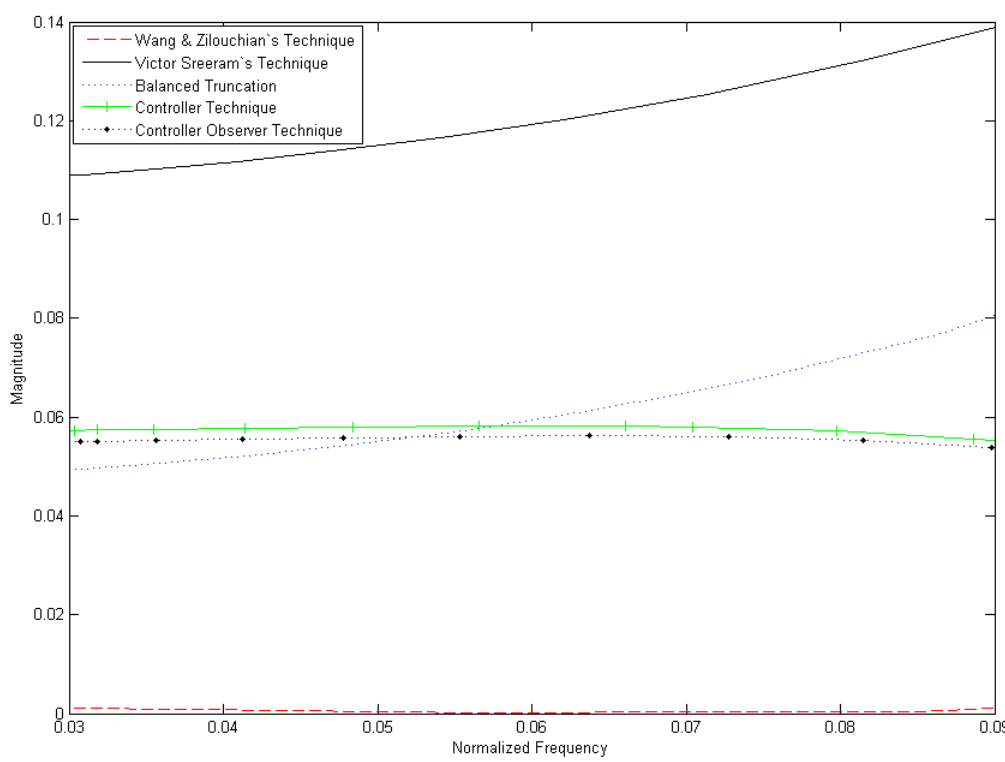

Figure 2.12: Error comparison - zoom-in view for 3^{rd} order LOAS

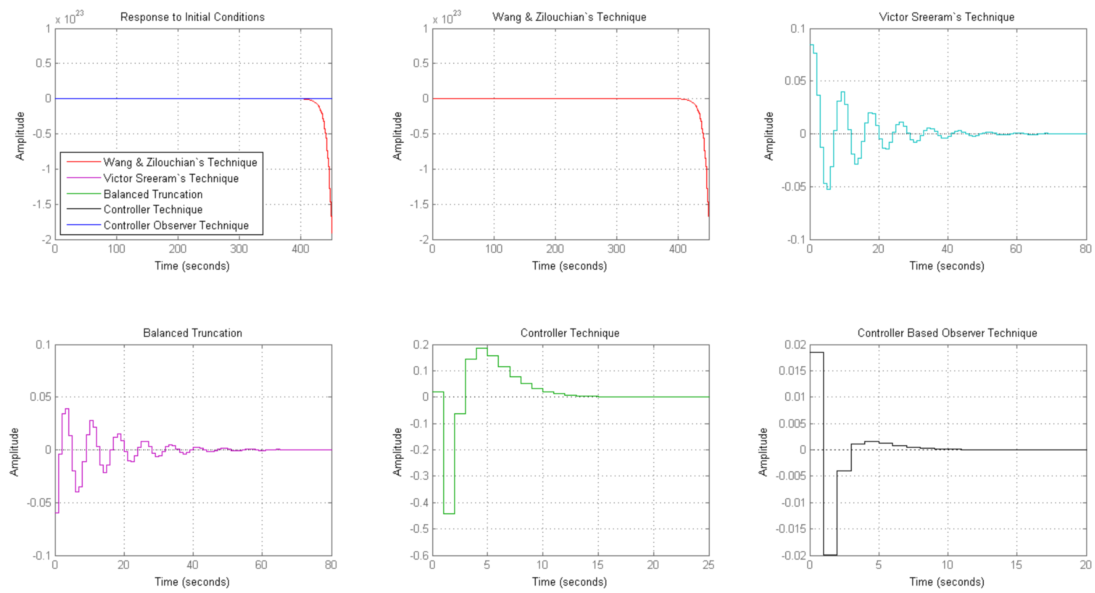

Figure 2.13: Natural response for 3^{rd} order LOAS

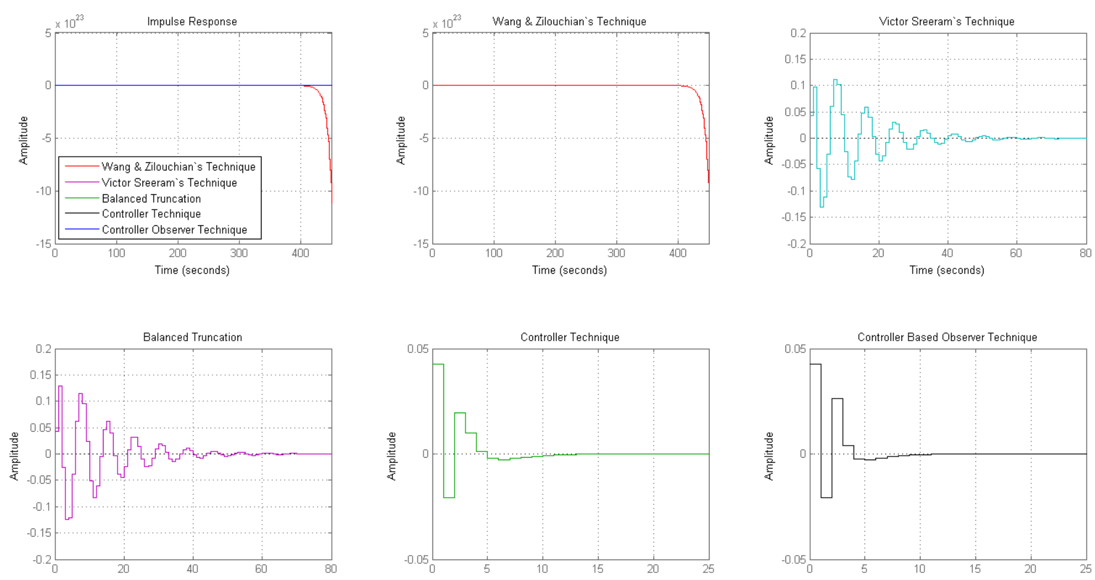

Figure 2.14: Impulse response for 3^{rd} order LOAS

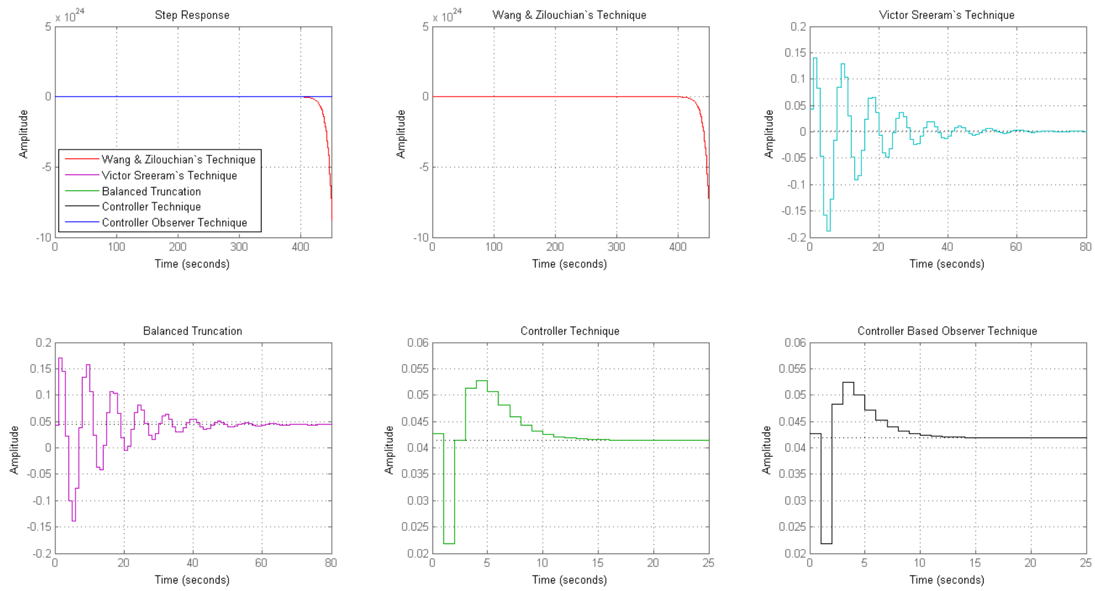

Figure 2.15: Step response for 3^{rd} order LOAS

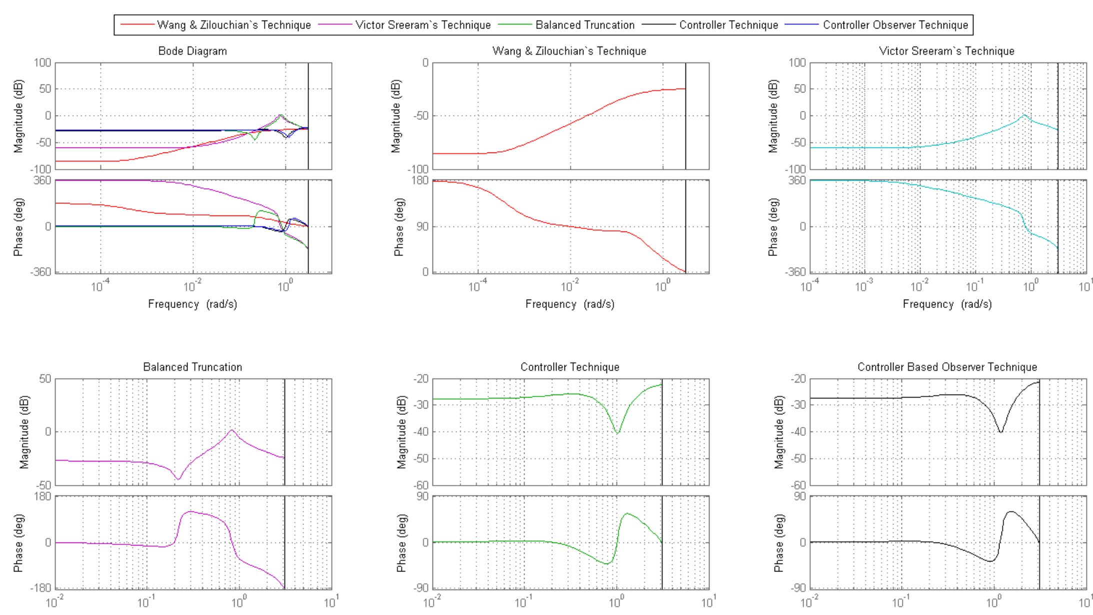

Figure 2.16: Bode plot for 3^{rd} order LOAS

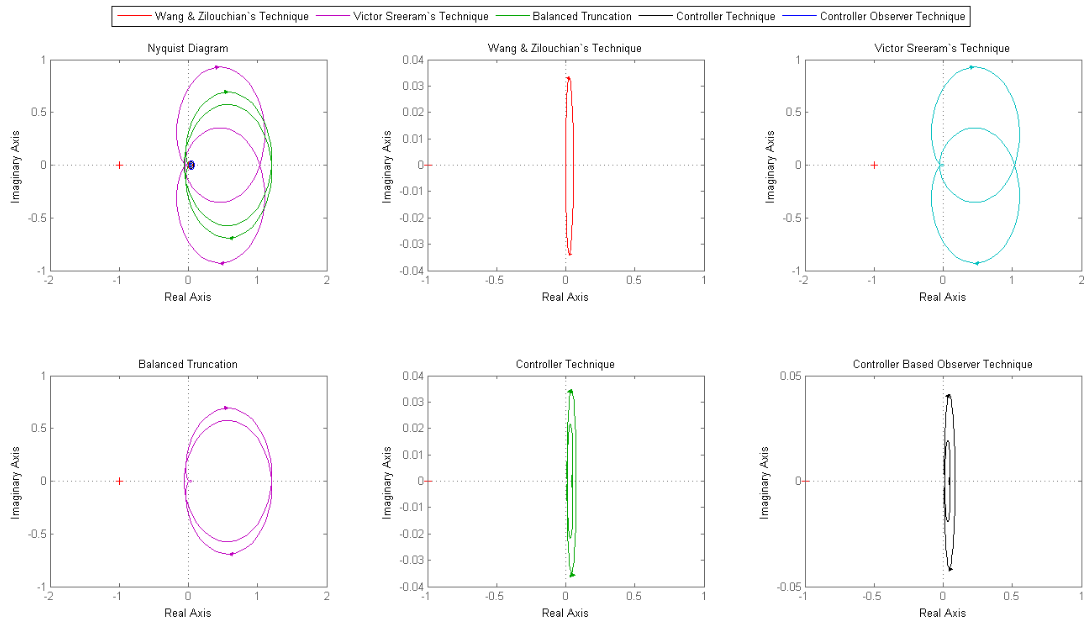

Figure 2.17: Nyquist plot for 3^{rd} order LOAS

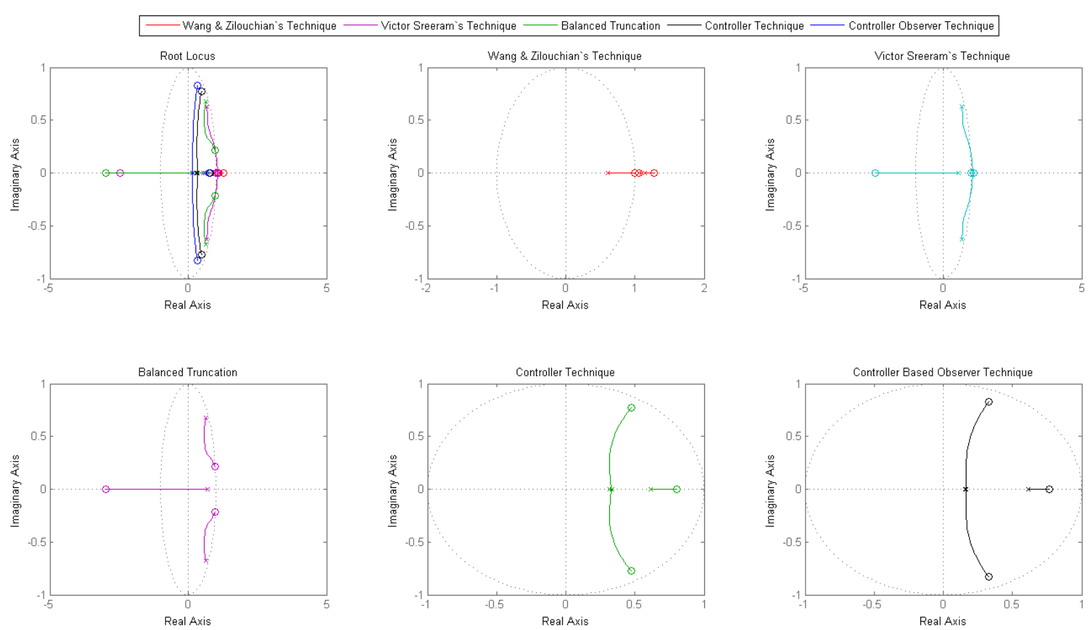

Figure 2.18: Root Locus plot for 3^{rd} order LOAS

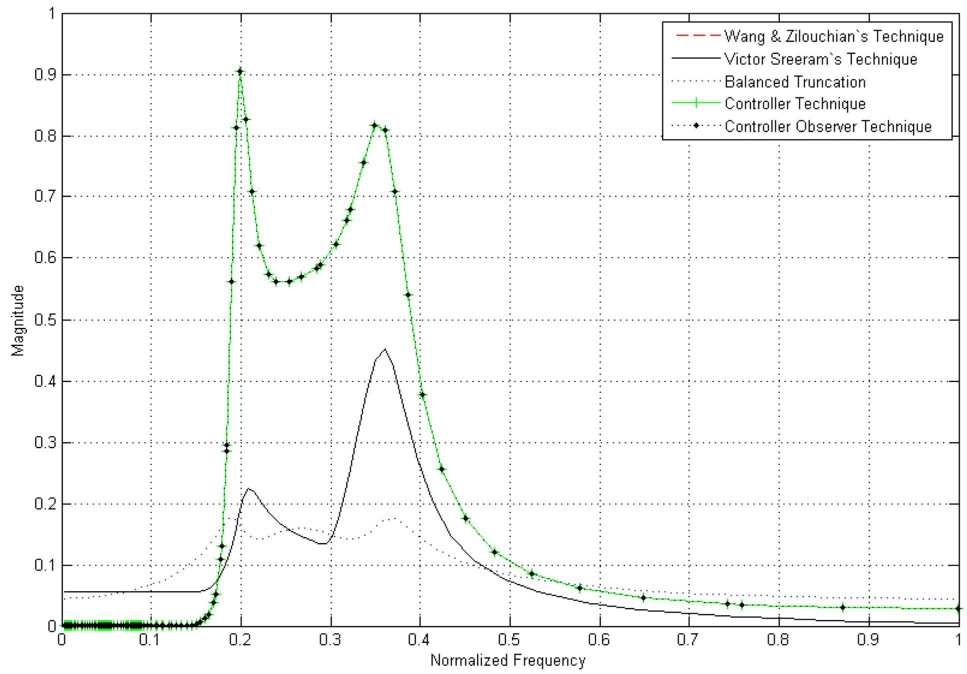

Figure 2.19: Error comparison for 5^{th} order LOAS

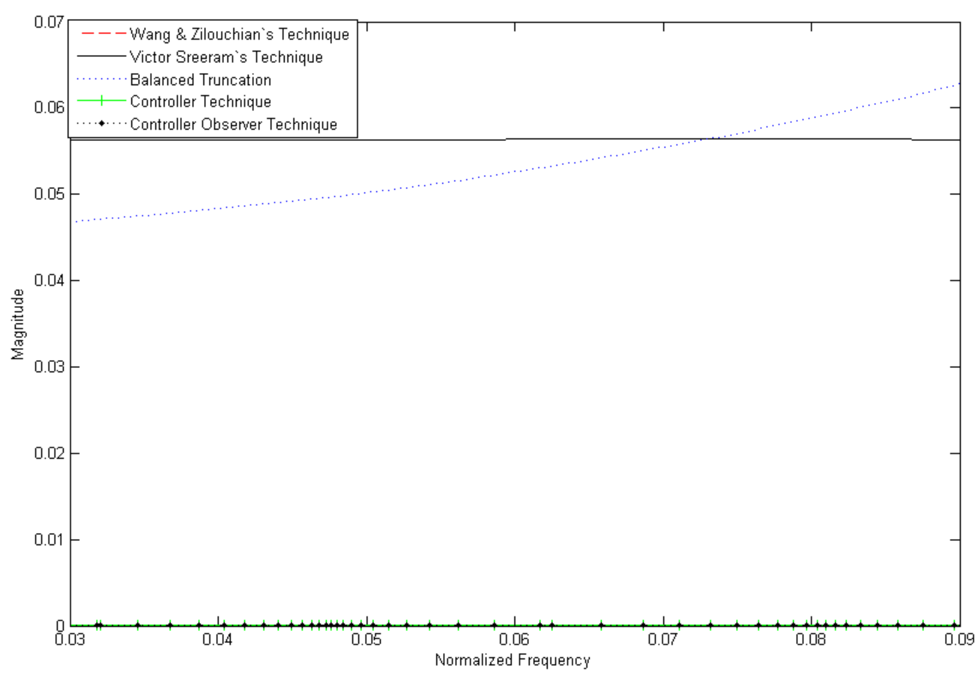

Figure 2.20: Error comparison - zoom-in view for 5^{th} order LOAS

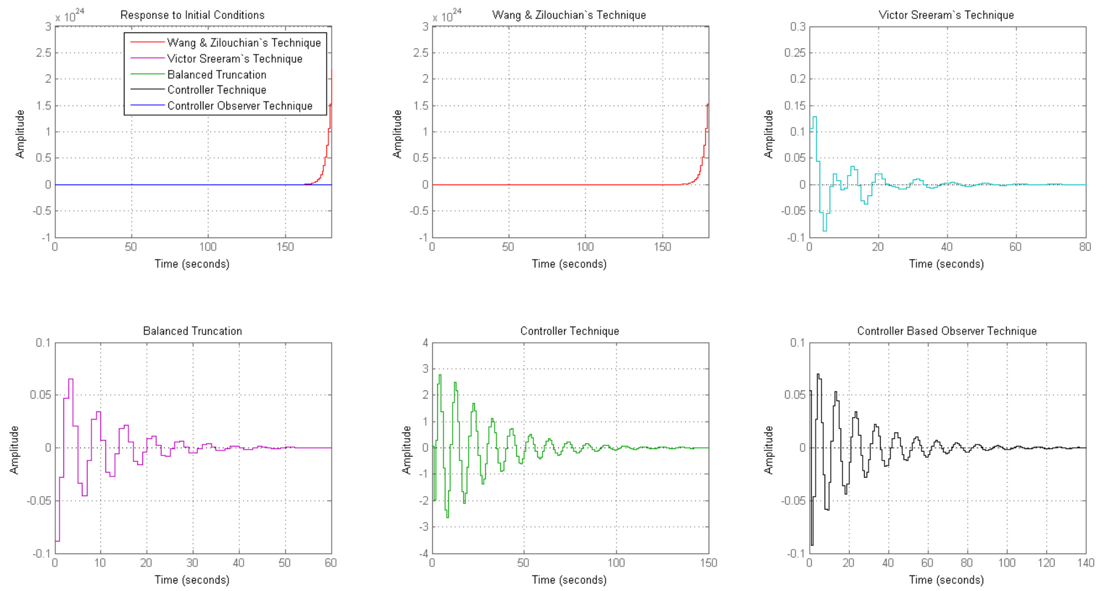

Figure 2.21: Natural response for 5^{th} order LOAS

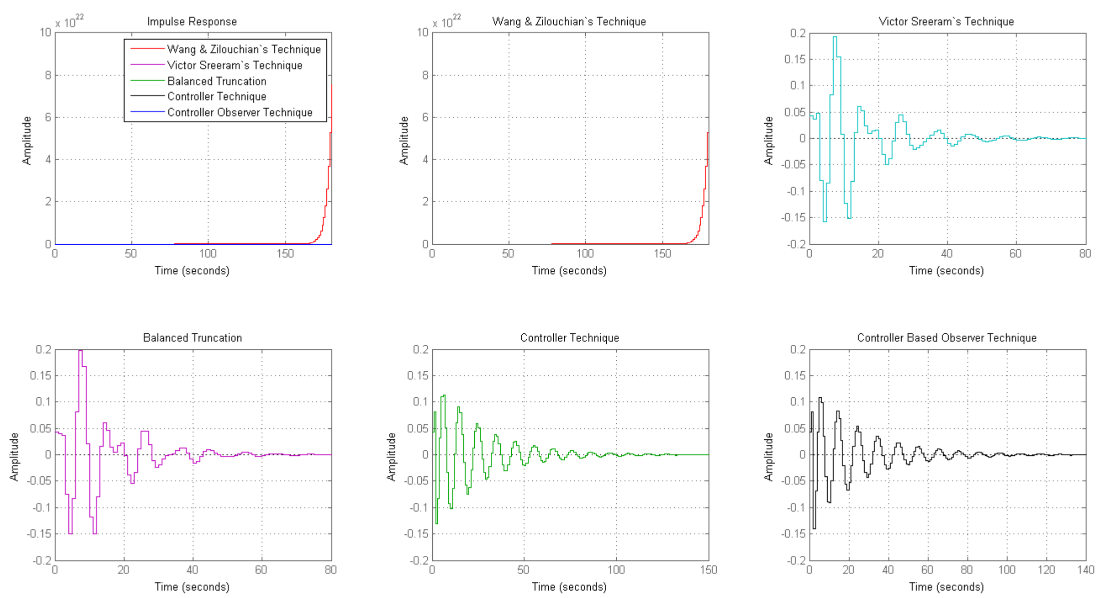

Figure 2.22: Impulse response for 5^{th} order LOAS

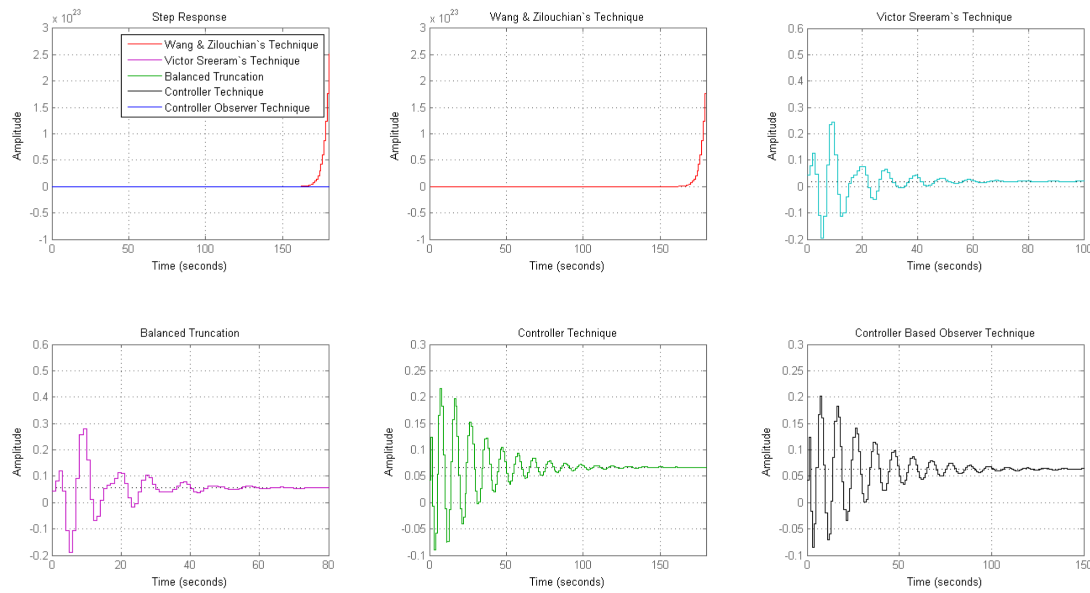

Figure 2.23: Step response for 5^{th} order LOAS

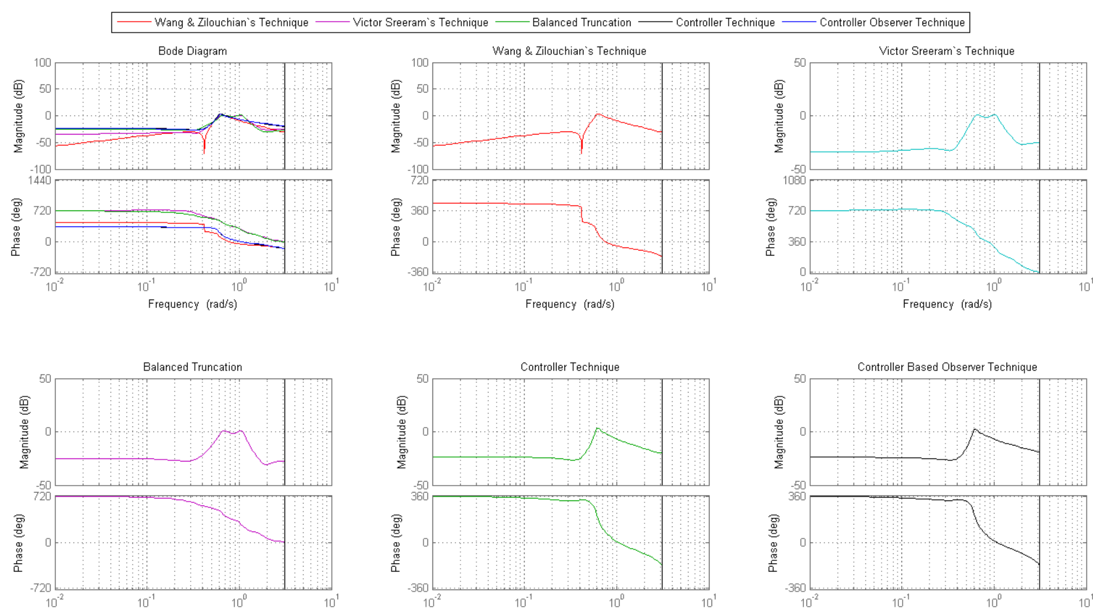

Figure 2.24: Bode plot for 5^{th} order LOAS

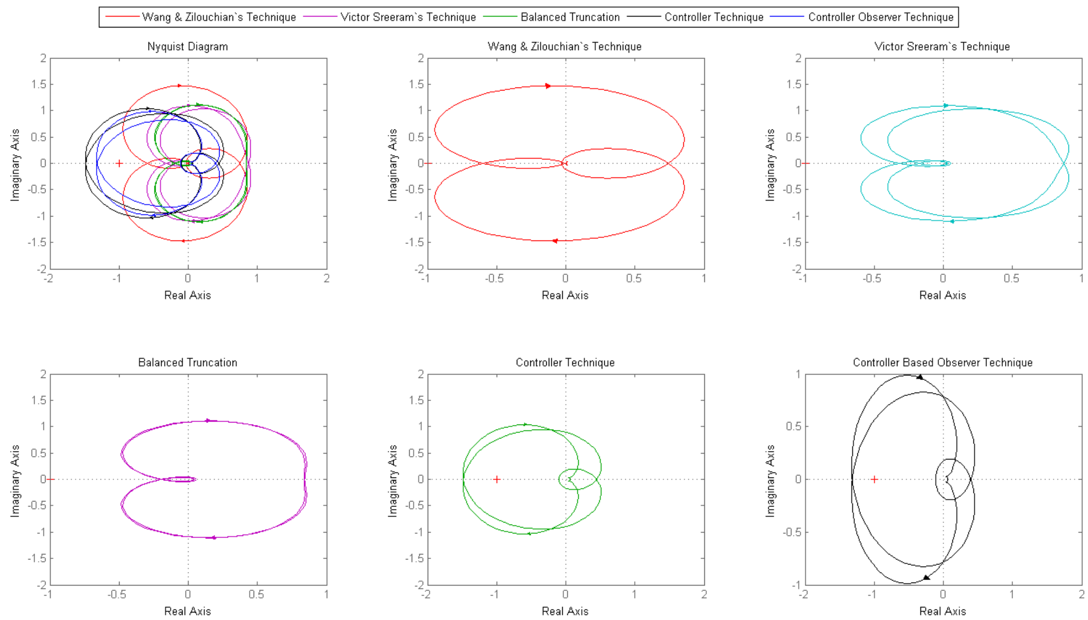

Figure 2.25: Nyquist plot for 5^{th} order LOAS

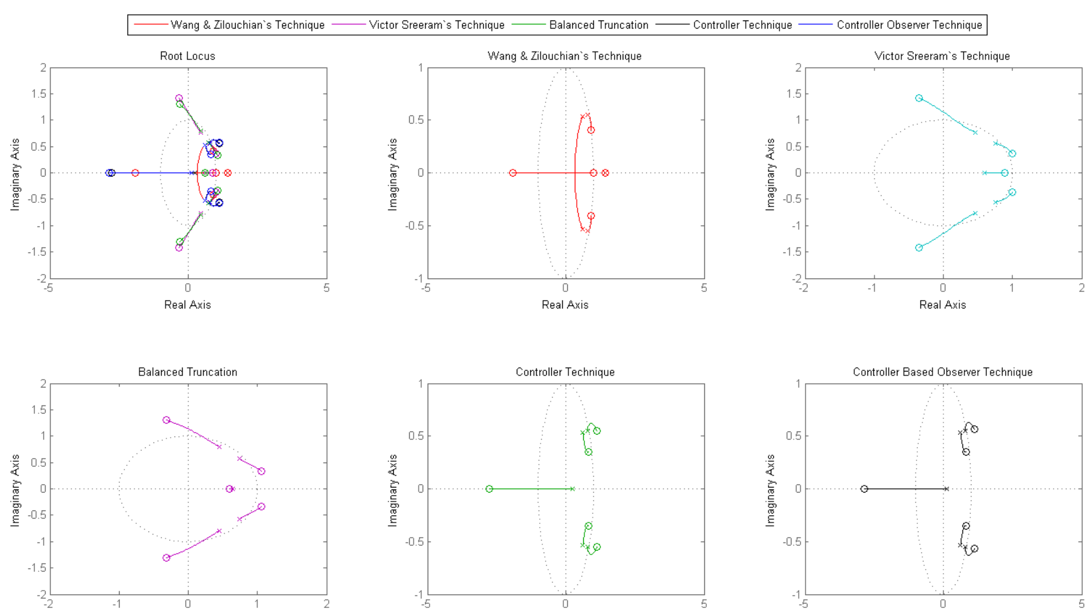

Figure 2.26: Root Locus plot for 5^{th} order LOAS

niques respectively. It can be orchestrated that the proposed technique gives low frequency response error. Simulation results show that GJ produces relatively less approximation error as compared to other existing techniques, but it yields sometime unstable LOAS as shown in Table 2.1, 2.2, 2.3 and 2.4. Proposed techniques produce less error for approximation (when compared with other preserving techniques for stability) and stable LOAS.

2.4 Conclusion

Improvement in frequency limited LOAS method for discrete systems is suggested. The suggested method provides stability of LOAS and projected error bounds. The error for approximation is less and comparable to other existing techniques for preserving stability. WZ's technique provide less error for approximation as compared to other persistent techniques, but it occasionally yield unstable LOAS.

Chapter 3

CONTROL DESIGN OF APPROXIMATED SYSTEMS USING FEEDBACK ANALYSIS OF STATES AND OUTPUT FOR CONTINUOUS TIME SYSTEMS

Lower Order Approximation Systems (LOAS) is a process of reducing the higher order system to lower order system. Higher order system involve lots of complexity for the analysis, design and its simulations. In control system theory it is very useful way to reduce the higher order model and perform tasks. It has several applications in control system theory [11]- [14].

One of the most widely used technique of LOAS is balance truncation [22]. It is used to obtained the LOAS. It provide the stable LOAS and does provide formula for error bound. It preserve the properties like input/output behaviour and passivity etc. While performing LOAS it use full frequency, however, it is not desired always to use ranges of complete frequency. This encourages to operate on frequency weights.

Enns [23] used this idea and announced the frequency weights (input, output and both sided) to obtain LOAS. It provides the LOAS which is stable in case of single sided weights, whereas for double sided weights it provide unstable LOAS. To solve this issue many other techniques have been presented in document [25]- [37].

Gawronski and Jaung (GJ) [38] pioneer of the limited frequency band Gramians based

LOAS, frequency weights were not used in this technique. Gramians are defined over limited frequency interval for continuous time system. It may cause unstable LOAS for stable original system and also not provide computable error bound formula. To solve this issue Gugercin and Antoulas (GA) [39], Victor Sreeram (VS) [35], Muhammad Imran (MI) [40] modified GJ's [38] technique and provide stable LOAS.

In this paper a new way is proposed which provide stability of LOAS and also provide the computable error bound formula. Results of Simulations explicitly present effectiveness and validity of the new proposed technique.

3.1 Preliminaries

Imagine a continuous time system which is stable

$$\dot{x} = \tilde{A}x(t) + \tilde{B}u(t) \tag{3.1}$$

$$y = \tilde{C}x(t) + \tilde{D}u(t)$$

$$\tilde{G}(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + D, \tag{3.2}$$

where $\tilde{A} \in R^{n\times n}$, $\tilde{B} \in R^{n\times m}$, $\tilde{C} \in R^{p\times n}$, $D \in R^{p\times m}$ and $\{\tilde{A}, \tilde{B}, \tilde{C}, D\}$ is its n^{th} order minimal realization.

$$\tilde{P}_c = \frac{1}{2\pi}\int_{-\infty}^{\infty}(j\omega I - \tilde{A})^{-1}\tilde{B}\tilde{B}^T(-j\omega I - \tilde{A}^T)^{-1}d\omega \tag{3.3}$$

$$\tilde{Q}_o = \frac{1}{2\pi}\int_{-\infty}^{\infty}(-j\omega I - \tilde{A}^T)^{-1}\tilde{C}^T\tilde{C}(j\omega I - \tilde{A})^{-1}d\omega \tag{3.4}$$

$\tilde{P}_c$ and $\tilde{Q}_o$ satisfy

$$\tilde{A}\tilde{P}_c + \tilde{P}_c\tilde{A}^T + \tilde{B}\tilde{B}^T = 0 \tag{3.5}$$

$$\tilde{A}^T\tilde{Q}_o + \tilde{Q}_o\tilde{A} + \tilde{C}^T\tilde{C} = 0 \tag{3.6}$$

By using Parseval's relationship

$$\begin{aligned}
\tilde{P}_c &= \frac{1}{2\pi}\int_{-\omega}^{+\omega}(j\omega I - \tilde{A})^{-1}\tilde{B}\tilde{B}^T(-j\omega I - \tilde{A}^T)^{-1}d\omega \\
\tilde{Q}_o &= \frac{1}{2\pi}\int_{-\omega}^{+\omega}(-j\omega I - \tilde{A}^T)^{-1}\tilde{C}^T\tilde{C}(j\omega I - \tilde{A})^{-1}d\omega
\end{aligned}$$

3.1.1 GJ's Technique [38]

$\tilde{P}_G$ and $\tilde{Q}_G$ can be defined as for limited interval

$\tilde{P}_G = \tilde{P}_c(\omega_2) - \tilde{P}_c(\omega_1)$, $\tilde{Q}_G = \tilde{Q}_o(\omega_2) - \tilde{Q}_o(\omega_1)$ $\tilde{P}_G$ and $\tilde{Q}_G$ satisfy

$$\tilde{A}\tilde{P}_G + \tilde{P}_G\tilde{A}^T + \tilde{B}\tilde{B}^T = 0 \tag{3.7}$$

$$\tilde{A}^T\tilde{Q}_G + \tilde{Q}_G\tilde{A} + \tilde{C}^T\tilde{C} = 0 \tag{3.8}$$

where

$$\tilde{X}_G = \left(\tilde{S}(\omega_2) - \tilde{S}(\omega_1)\right)\tilde{B}\tilde{B}^T + \tilde{B}\tilde{B}^T\left(\tilde{S}^*(\omega_2) - \tilde{S}^*(\omega_1)\right)$$

$$\tilde{X}_G = U\begin{bmatrix} S_{G1} & 0 \\ 0 & S_{G2} \end{bmatrix}U^T$$

$$\tilde{Y}_G = \left(\tilde{S}^*(\omega_2) - S^*(\omega_1)\right)\tilde{C}^T\tilde{C} + \tilde{C}^T\tilde{C}\left(\tilde{S}(\omega_2) - S(\omega_1)\right)$$

$$\tilde{Y}_G = V\begin{bmatrix} R_{G1} & 0 \\ 0 & R_{G2} \end{bmatrix}V^T$$

$$\bar{S}(\omega) = \frac{j}{2\pi}\ln\left(\left(j\omega I + \tilde{A}\right)\left(-j\omega I + \tilde{A}\right)^{-1}\right)$$

$$S_{G1} = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ 0 & 0 & \cdots & s_l \end{bmatrix}, \quad S_{G2} = \begin{bmatrix} s_{l+1} & 0 & \cdots & 0 \\ 0 & s_{l+2} & \cdots & 0 \\ 0 & 0 & \cdots & s_n \end{bmatrix},$$

$$R_{G1} = \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ 0 & 0 & \cdots & r_k \end{bmatrix}, \quad R_{G2} = \begin{bmatrix} r_{k+1} & 0 & \cdots & 0 \\ 0 & r_{k+2} & \cdots & 0 \\ 0 & 0 & \cdots & r_n \end{bmatrix}$$

l and k are eigenvalues which is positive of $\tilde{X}_G$ and $\tilde{Y}_G$ respectively. $\tilde{S}^*(\omega)$ is conjugate transpose of $\tilde{S}(\omega)$. Let

$$T_G^T\tilde{Q}_GT_G = T_G^{-1}\tilde{P}_GT_G^{-T} = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$$

and LOAS obtained is

$$G_r(s) = C_r(sI - A_r)^{-1}B_r + D_r, \tag{3.9}$$

r is the order of LOAS.

$$T_G^{-1}\tilde{A}T_G = \begin{bmatrix} A_r & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad T_G^{-1}\tilde{B} = \begin{bmatrix} B_r \\ B_2 \end{bmatrix} \tag{3.10}$$

$$\tilde{C}T_G = \begin{bmatrix} C_r & C_2 \end{bmatrix}, \qquad D \tag{3.11}$$

where $\sigma_j \geq \sigma_{j+1},\ j = 1, 2, 3, \ldots, n-1$, $\sigma_r > \sigma_{r+1}$ and T_G is used to transform the actual system. Partitioning transformed system to obtained LOAS.

Remark 8 *GJ [38] sometimes produce unstable LOAS because input/output related matrices X_G and Y_Ge may be negative-definite or negative semi-definite.*

3.1.2 Existing Stability Preserving Frequency Limited Techniques

Let P_E and Q_E satisfy

$$AP_E + P_EA^T + B_EB_E^T = 0 \tag{3.12}$$

$$A^TQ_E + Q_EA + C_E^TC_E = 0 \tag{3.13}$$

$$B_A = U \begin{bmatrix} S_{G1}^{1/2} & 0 \\ 0 & |S_{G2}|^{1/2} \end{bmatrix}$$

$$B_S = U \begin{bmatrix} S_{G1}^{1/2} & 0 \\ 0 & 0 \end{bmatrix}$$

$$B_I = \begin{cases} U(S - s_n I)^{1/2} & \text{for } s_n < 0 \\ US^{1/2} & \text{for } s_n \geq 0 \end{cases}$$

$$C_A = \begin{bmatrix} R_{G1}^{1/2} & 0 \\ 0 & |R_{G2}|^{1/2} \end{bmatrix} V^T$$

$$C_S = \begin{bmatrix} R_{G1}^{1/2} & 0 \\ 0 & 0 \end{bmatrix} V^T$$

$$C_I = \begin{cases} (R - r_n I)^{1/2} V^T & \text{for } r_n < 0 \\ R^{1/2} V^T & \text{for } r_n \geq 0. \end{cases}$$

Let a transformation matrix T_E is obtained as:

$$T_E^T Q_E T_E = T_E^{-1} P_E T_E^{-T} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}$$

The LOAS are obtained after transforming the original system using the matrix T_E similar to equations $(9) - (10)$. These LOAS are guaranteed stable and error bounds also exist.

The existing stability preserving techniques [39], [35], [40] modified $\tilde{X}_G$ and $\tilde{Y}_G$ to ensure positive/semipositive definite.

Let new virtual/fictitious controllability P_v and observability Q_v Gramians are computed as

$$\tilde{A}P_v + P_v\tilde{A}^T + B_vB_v^T = 0 \tag{3.14}$$

$$\tilde{A}^TQ_v + \tilde{A}Q_v + C_v^TC_v = 0 \tag{3.15}$$

The new virtual input and output matrices respectively are given as B_v and C_v where

$$\hat{s} = \sum_{i=l+1}^{n} s_i \ , \quad \hat{r} = \sum_{i=k+1}^{n} r_i$$

$$B_v = \begin{cases} U(S - \hat{s}I)^{1/2} & \text{for } s_n < 0 \\ US^{1/2} & \text{for } s_n \geq 0 \end{cases}$$

$$C_v = \begin{cases} (R - \hat{r}I)^{1/2}V^T & \text{for } r_n < 0 \\ R^{1/2}V^T & \text{for } r_n \geq 0. \end{cases}$$

Let similarity transformation matrix T_v is calculated as

$$T_v^TQ_vT_v = T_v^{-1}P_vT_v^{-T} = diag\{\sigma_1, \sigma_2, \sigma_3 \ldots, \sigma_n\}$$

where $\sigma_j \geq \sigma_{j+1}$ and $\sigma_r \geq \sigma_{r+1}$. The LOAS are obtained by transforming and partitioning the similar way of equation $(9) - (10)$.

Remark 9 *Since $X_G \leq B_vB_v^T \geq 0$, $Y_G \leq C_v^TC_v \geq 0$, $P_v > 0$ and $Q_v > 0$. Which results, minimality of the realization $(\tilde{A}, B_v, C_v)$ and stability of the LOAS is guaranteed. The stability of LOAS follows from [22] balanced truncation.*

Theorem 4 *Let* $rank\begin{bmatrix} B_v & \tilde{B}\end{bmatrix} = rank\,[B_v]$ *and* $rank\begin{bmatrix} C_v \\ \tilde{C}\end{bmatrix} = rank\,[C_v]$ *following error bound holds*

$$\|\tilde{G}(s) - G_r(s)\|_\infty \leq 2\|L_v\|\|K_v\|\sum_{j=r+1}^{n}\sigma_j$$

where

$$L_v = \begin{cases} CV(R - \hat{r}I)^{-1/2} & \textit{for } r_n < 0 \\ CVR^{-1/2} & \textit{for } r_n \geq 0 \end{cases}$$

$$K_v = \begin{cases} (S - \hat{s}I)^{-1/2}U^T B & \textit{for } s_n < 0 \\ S^{-1/2}U^T B & \textit{for } s_n \geq 0 \end{cases}$$

Proof: Since $rank\begin{bmatrix} B_v & \tilde{B}\end{bmatrix} = rank\,[B_v]$ and $rank\begin{bmatrix} C_v \\ \tilde{C}\end{bmatrix} = rank\,[C_v]$, the relationships $\tilde{B} = B_v K_v$ and $\tilde{C} = L_v C_v$ hold. By partitioning $B_v = \begin{bmatrix} B_{v1} \\ B_{v2}\end{bmatrix}$, $C_v = \begin{bmatrix} C_{v1} & C_{v2}\end{bmatrix}$ and substituting $B_r = B_{v1}K_v$, $C_r = L_vC_{v1}$ respectively yields

$$\begin{aligned}
\|\tilde{G}(s) - G_r(s)\|_\infty &= \|\tilde{C}(sI-\tilde{A})^{-1}\tilde{B}-C_r(sI-A_r)^{-1}B_r\|_\infty \\
&= \|L_vC_v(sI-A)^{-1}B_vK_v \\
&\quad -L_vC_{v1}(sI-A_r)^{-1}B_{v1}K_v\|_\infty \\
&= \|L_v(C_v(sI-\tilde{A})^{-1}B_v \\
&\quad -C_{v1}(sI-A_r)^{-1}B_{v1})K_v\|_\infty \\
&\leq \|L_v\|\|(C_v(sI-\tilde{A})^{-1}B_v \\
&\quad -C_{v1}(sI-A_r)^{-1}B_{v1})\|_\infty\|K_v\|
\end{aligned}$$

If $\{A_r, B_{v1}, C_{v1}\}$ is model obtained after reduction of original system $\{\tilde{A}, B_v, C_v\}$.

$$\|(C_v(sI-A)^{-1}B_v - C_{v1}(sI-A_r)^{-1}B_{v1})\|_\infty \leq 2\sum_{j=r+1}^{n}\sigma_j.$$

Therefore,

$$\|G(s) - G_r(s)\|_\infty \leq 2\|L_v\|\|K_v\| \sum_{j=r+1}^{n} \sigma_j$$

Remark 10 *The rank condition follows from [35].*

3.2 Main Results

The existing stability preserving techniques [39], [35], [40] modified X_G and Y_G to ensure positive/semipositive definite.

Let new virtual/fictitious controllability P_{EI} and observability Q_{EI} Gramians are computed as

$$\tilde{A}P_{EI} + P_{EI}\tilde{A}^T + B_{EI}B_{EI}^T = 0 \tag{3.16}$$

$$\tilde{A}^T Q_{EI} + \tilde{A}Q_{EI} + C_{EI}^T C_{EI} = 0 \tag{3.17}$$

The new virtual matrices be given as B_{EI} and C_{EI} where Let similarity transformation matrix T_{EI} is calculated as

$$T_{EI}^T Q_{EI} T_{EI} = T_{EI}^{-1} P_{EI} T_{EI}^{-T} = diag\{\sigma_1, \sigma_2, \sigma_3 \ldots, \sigma_n\}$$

where $\sigma_j \geq \sigma_{j+1}$ and $\sigma_r \geq \sigma_{r+1}$. The LOAS are obtained by transforming and partitioning the similar way of equation $(9) - (10)$.

Remark 11 *Since $X_G = B_{EI}B_{EI}^T \leq 0$, $Y_G = C_{EI}^T C_{EI} \leq 0$, $P_{EI} > 0$ and $Q_{EI} > 0$. Which results, minimality of the realization $(\tilde{A}, B_{EI}, C_{EI})$ however, stability of the LOAS is not guaranteed.*

Design of Feedback Controller for Lower Order Approximated System:

Let LOAS,

$$\begin{aligned} \dot{x}_r(t) &= A_r x_r(t) + B_r u(t) \\ y_r(t) &= C_r x_r(t) + D_r u(t) \end{aligned}$$

$$G_r(s) = C_r(sI - A_r)^{-1}B_r + D_r,$$

is obtained by approximating the actual system (in the desired limited frequency range

$[\omega_1, \omega_2]$) where $\omega_2 > \omega_1$, where $\{A_r \in R^{r\times r}, B_r \in R^{r\times m}, C_r \in R^{p\times r}, D_r \in R^{p\times m}\}$ with $r < n$.

To make LOAS stable introducing State Feedback Controller and Output Feedback Controller with states Feedback.

using unstable lower order approximated system as given following,

$$\begin{aligned} \dot{x_r}(t) &= A_r x_r(t) + B_r u(t) \\ y_r(t) &= C_r x_r(t) + D_r u(t) \end{aligned}$$

$$G_r(s) = C_r(sI - A_r)^{-1}B_r + D_r,$$

lower order control law is applying in above system $u(t) = -K_r x_r + \hat{u}$ to feedback its lower order states to stabilizing the lower order approximated system which is given following,

$$\begin{aligned} \dot{x_r}(t) &= A_r x_r(t) + B_r(-K_r x_r + \hat{u}) \\ y_r(t) &= C_r x_r(t) + D_r(-K_r x_r + \hat{u}) \end{aligned}$$

$$\begin{aligned} \dot{x_r}(t) &= (A_r - B_r K_r)x_r + B_r\hat{u} \\ y_r(t) &= (C_r - D_r K_r)x_r + D_r\hat{u} \end{aligned}$$

$$G_r(s) = (C_r - D_r K_r)(sI - (A_r - B_r K_r))^{-1}B_r + D_r,$$

where $\{(A_r - B_r K_r) = \hat{A}_r \in R^{r\times r}, B_r = \hat{B}_r \in R^{r\times m}, (C_r - D_r K_r) = \hat{C}_r \in R^{p\times r}, D_r =$

$\hat{D}_r \in R^{p\times m}\}$ with $r < n$. Stable lower order approximated system will be given as,

$$\begin{aligned} \dot{\hat{x}}_r(t) &= \hat{A}_r x_r + \hat{B}_r \hat{u} \\ \hat{y}_r(t) &= \hat{C}_r x_r + \hat{D}_r \hat{u} \end{aligned} \tag{3.18}$$

$$\hat{G}_r(s) = \hat{C}_r(sI - \hat{A}_r)^{-1}\hat{B}_r + \hat{D}_r, \tag{3.19}$$

Lower Order Approximated System is also give in Fig. 3.1

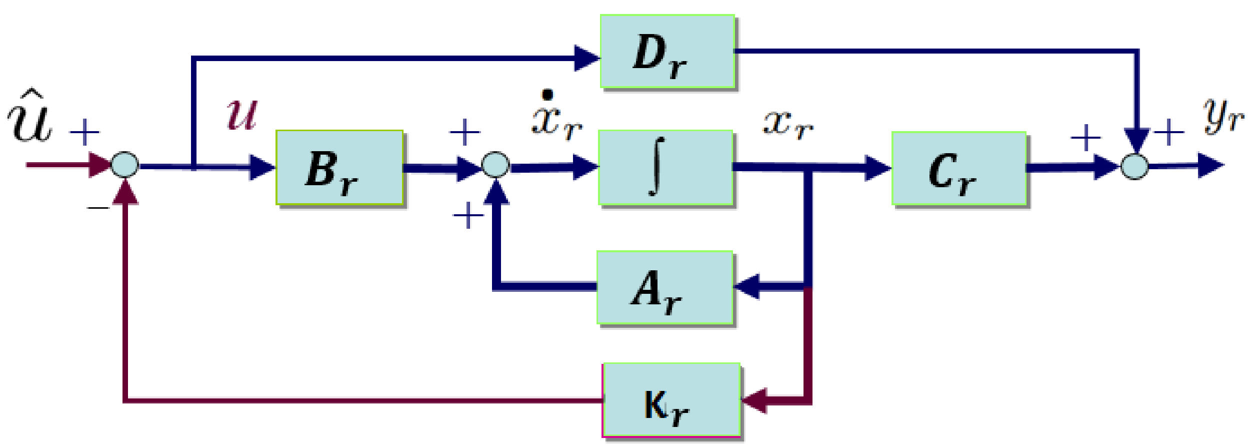

Figure 3.1: State Feedback Control for Lower Order Approximated System

Remark 12 *Since* $X_{GJ} = B_{EI}B_{EI}^T \leq 0$, $Y_{GJ} = C_{EI}^T C_{EI} \leq 0$, $P_{EI} \leq 0$ *and* $Q_{EI} \leq 0$. *Therefore, the realization* (A, B_{EI}, C_{EI}) *is not stable, however,* $(\hat{A}_r, \hat{B}_r, \hat{C}_r)$ *and stability of lower order approximated system is guaranteed given in Fig. 3.1.*

Design of Observer Based Feedback Controller for Lower Order Approximated System:

Let LOAS,

$$\dot{x_r}(t) = A_r x_r(t) + B_r u(t)$$

$$y_r(t) = C_r x_r(t) + D_r u(t)$$

$$G_r(s) = C_r(sI - A_r)^{-1}B_r + D_r,$$

is obtained by approximating the actual system (in the desired limited frequency range $[\omega_1, \omega_2]$) where $\omega_2 > \omega_1$, where $\{A_r \in R^{r\times r}, B_r \in R^{r\times m}, C_r \in R^{p\times r}, D_r \in R^{p\times m}\}$ with $r < n$.

To make LOAS stable introducing Observer Based State Feedback Controller.

using unstable lower order approximated system as given following,

$$\dot{x_r}(t) = A_r x_r(t) + B_r u(t)$$

$$y_r(t) = C_r x_r(t) + D_r u(t)$$

$$G_r(s) = C_r(sI - A_r)^{-1}B_r + D_r,$$

let,

Lower Order Approximated System:

$$\dot{x_r}(t) = A_r x_r(t) + B_r u(t)$$

$$y_r(t) = C_r x_r(t) + D_r u(t)$$

Lower Order Approximated Observer:

$$\begin{aligned}\dot{\hat{x}}_r(t) &= A_r\hat{x}_r(t) + L(y_r - C_r\hat{x}_r) + B_r u(t)\\ \hat{y}_t(k) &= C_r\hat{x}_r(t) + D_r u(t)\end{aligned}$$

State Feedback Law:

$u(t) = -K_r x_r + \hat{u}$

lower order control law is applying in above system $u(t) = -K_r x_r + \hat{u}$ to feedback its observer based lower order states to stabilizing the lower order approximated system which is given following,

$$\begin{aligned}\dot{\hat{x}}_r(t) &= (A_r - LC_r)\hat{x}_r(k) + LC_r x_r + B_r(-K_r\hat{x}_r + \hat{u})\\ \hat{y}_r(t) &= C_r\hat{x}_r(t) + D_r(-K_r\hat{x}_r + \hat{u})\end{aligned}$$

$$\begin{aligned}\dot{\hat{x}}_r(t) &= (A_r - LC_r - B_rK_r)\hat{x}_r + B_r\hat{u}\\ \hat{y}_r(t) &= (C_r - D_rK_r)\hat{x}_r + D_r\hat{u}\end{aligned}$$

$$G_r(s) = (C_r - D_rK_r)(sI - (A_r - LC_r - B_rK_r))^{-1}B_r + D_r,$$

where $\{(A_r - LC_r - B_rK_r) = \hat{A_{ro}} \in R^{r\times r}, B_r = \hat{B_{ro}} \in R^{r\times m}, (C_r - D_rK_r) = \hat{C_{ro}} \in R^{p\times r}, D_r = \hat{D_r}o \in R^{p\times m}\}$ with $r < n$. Stable lower order approximated system will be given as,

$$\begin{aligned}\dot{\hat{x}}_r(t) &= \hat{A_{ro}}x_r + \hat{B_{ro}}\hat{u} \qquad (3.20)\\ \hat{y}_r(t) &= \hat{C_{ro}}x_r + \hat{D_r}o\hat{u}\end{aligned}$$

$$\hat{G_{ro}}(s) = \hat{C_{ro}}(sI - \hat{A_{ro}})^{-1}\hat{B_{ro}} + \hat{D_{ro}}, \tag{3.21}$$

Observer Based Lower Order Approximated System is also give in Fig. 3.2

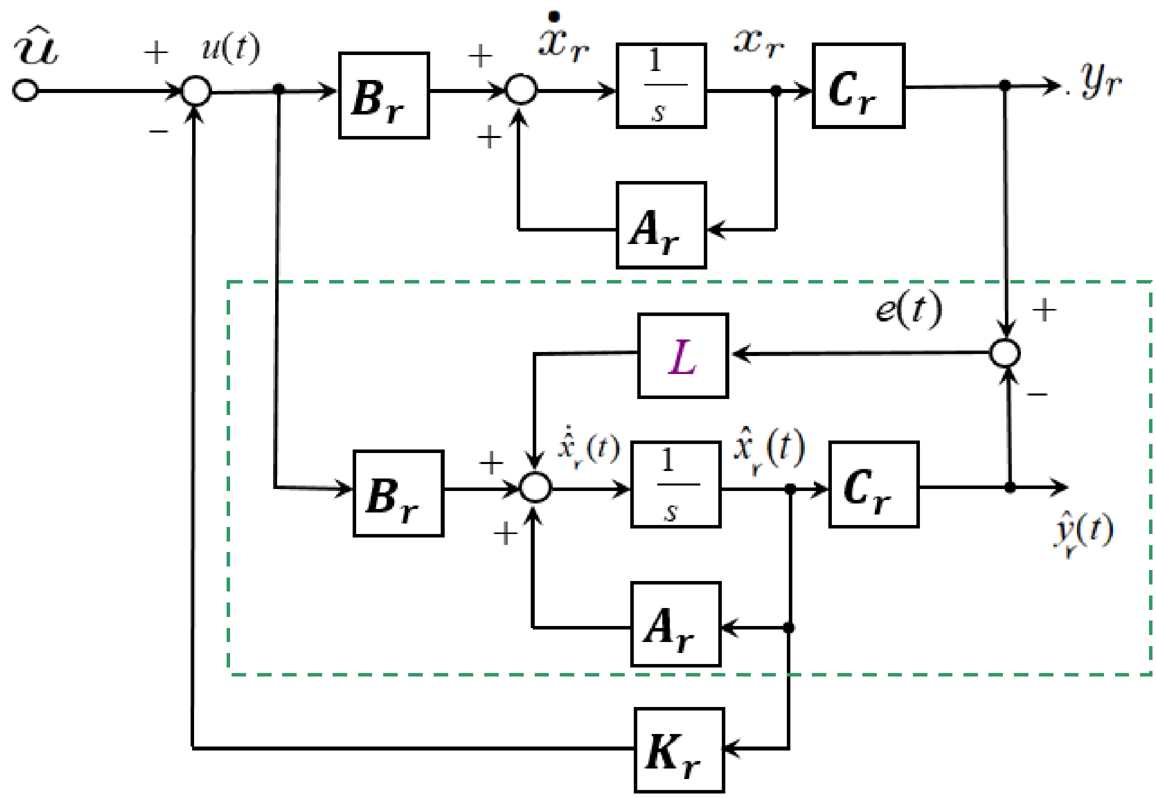

Figure 3.2: Observer Based State Feedback Control for Lower Order Approximated System

Theorem 5 *Let* $rank \begin{bmatrix} B_{EI} & \tilde{B} \end{bmatrix} = rank\, [B_{EI}]$ *and* $rank \begin{bmatrix} C_{EI} \\ \tilde{C} \end{bmatrix} = rank\, [C_{EI}]$ *following error bound holds*

$$\|\tilde{G}(s) - \hat{G}_r(s)\|_\infty \leq 2\|L_{EI}\|\|K_{EI}\| \sum_{j=r+1}^{n} \sigma_j$$

where

$$L_{EI} = \left\{ \; CVR_W^{-1/2} \right.$$
$$K_{EI} = \left\{ \; S_W^{-1/2}U^T B \right.$$

*Proof:*Since $rank\begin{bmatrix}B_{EI} & \tilde{B}\end{bmatrix} = rank\,[B_{EI}]$ and $rank\begin{bmatrix} C_{EI} \\ \tilde{C}\end{bmatrix} = rank\,[C_{EI}]$, the relationships $B = B_{EI}K_{EI}$ and $C = L_{EI}C_{EI}$ hold. By partitioning $B_{EI} = \begin{bmatrix} B_{EI_1} \\ B_{EI_2}\end{bmatrix}, C_{EI} = \begin{bmatrix} C_{EI_1} & C_{EI_2}\end{bmatrix}$ and substituting $\hat{B}_r = B_{EI_1}K_{EI}$, $\hat{C}_r = L_{EI}C_{EI_1}$ respectively yields

$$\begin{aligned}
\|\tilde{G}(s) - \hat{G}_r(s)\|_\infty &= \|\tilde{C}(sI-A)^{-1}\tilde{B}-\hat{C}_r(sI-\hat{A}_r)^{-1}\hat{B}_r\|_\infty \\
&= \|L_{EI}C_{EI}(sI-A)^{-1}B_{EI}K_{EI} \\
&\quad -L_{EI}C_{EI_1}(sI-\hat{A}_r)^{-1}B_{EI_1}K_{EI}\|_\infty \\
&= \|L_{EI}(C_{EI}(sI-A)^{-1}B_{EI} \\
&\quad -C_{EI_1}(sI-\hat{A}_r)^{-1}B_{EI_1})K_{EI}\|_\infty \\
&\le \|L_{EI}\|\|(C_{EI}(sI-A)^{-1}B_{EI} \\
&\quad -C_{EI_1}(sI-\hat{A}_r)^{-1}B_{EI_1})\|_\infty\|K_{EI}\|
\end{aligned}$$

If $\{\hat{A}_r, B_{EI_1}, C_{EI_1}\}$ is LOAS of original system $\{A, B_{EI}, C_{EI}\}$.

$$\|(C_{EI}(sI-A)^{-1}B_{EI} - C_{EI_1}(sI-\hat{A}_r)^{-1}B_{EI_1})\|_\infty \le 2\sum_{j=r+1}^{n}\sigma_j.$$

Therefore,

$$\|\tilde{G}(s) - \hat{G}_r(s)\|_\infty \leq 2\|L_{EI}\|\|K_{EI}\| \sum_{j=r+1}^{n} \sigma_j$$

Theorem 6 *Let* $rank\,[B_{EI}\ \ B] = rank\,[B_{EI}]$ *and* $rank \begin{bmatrix} C_{EI} \\ C \end{bmatrix} = rank\,[C_{EI}]$ *following error bound holds*

$$\|G(s) - \hat{G}_r(s)\|_\infty \leq 2\|L_{EI}\|\|K_{EI}\| \sum_{j=r+1}^{n} \sigma_j$$

where

$$L_{EI} = \left\{ \ CVR_W^{-1/2} \right.$$
$$K_{EI} = \left\{ \ S_W^{-1/2} U^T B \right.$$

Proof: Since $rank\,[B_{EI}\ \ B] = rank\,[B_{EI}]$ and $rank \begin{bmatrix} C_{EI} \\ C \end{bmatrix} = rank\,[C_{EI}]$, the relationships $B = B_{EI}K_{EI}$ and $C = L_{EI}C_{EI}$ hold. By partitioning $B_{EI} = \begin{bmatrix} B_{EI_1} \\ B_{EI_2} \end{bmatrix}, C_{EI} =$

$\begin{bmatrix} C_{EI_1} & C_{EI_2} \end{bmatrix}$ and substituting $\hat{B_r o} = B_{EI_1} K_{EI}$, $\hat{C_r o} = L_{EI} C_{EI_1}$ respectively yields

$$\begin{aligned} \|G(s) - \hat{G_r o}(s)\|_\infty &= \|C(sI-A)^{-1}B - \hat{C_r o}(sI - \hat{A_{ro}})^{-1}\hat{B_r o}\|_\infty \\ &= \|L_{EI}C_{EI}(sI-A)^{-1}B_{EI}K_{EI} \\ &\quad -L_{EI}C_{EI_1}(sI-\hat{A_{ro}})^{-1}B_{EI_1}K_{EI}\|_\infty \\ &= \|L_{EI}(C_{EI}(sI-A)^{-1}B_{EI} \\ &\quad -C_{EI_1}(sI-\hat{A_{ro}})^{-1}B_{EI_1})K_{EI}\|_\infty \\ &\leq \|L_{EI}\|\|(C_{EI}(sI-A)^{-1}B_{EI} \\ &\quad -C_{EI_1}(sI-\hat{A_{ro}})^{-1}B_{EI_1})\|_\infty\|K_{EI}\| \end{aligned}$$

If $\{\hat{A_{ro}}, B_{EI_1}, C_{EI_1}\}$ is LOAS given of original system $\{A, B_{EI}, C_{EI}\}$.

$$\|(C_{EI}(sI-A)^{-1}B_{EI} - C_{EI_1}(sI-\hat{A_{ro}})^{-1}B_{EI_1})\|_\infty \leq 2\sum_{j=r+1}^{n}\sigma_j.$$

Therefore,

$$\|G(s) - \hat{G_r o}(s)\|_\infty \leq 2\|L_{EI}\|\|K_{EI}\|\sum_{j=r+1}^{n}\sigma_j$$

Remark 13 *For $X_G \geq 0$, $Y_G \geq 0$ results $P_G = P_E = P_{EI}$ and $Q_G = Q_E = Q_{EI}$. Otherwise $P_G < P_{EI}$, $Q_G < Q_{EI}$. Which results Hankel values for frequency limited satisfy : $(\lambda_j[P_G Q_G])^{1/2} \leq (\lambda_j[P_{EI} Q_{EI}])^{1/2}$*

Table 3.1: LOAS for 1^{st} order

Techniques	LOAS
Gawronski and Jaung	$\frac{s-0.2044}{s-6.454}$
Gugercin and Antaulos	$\frac{s+3.102}{s+2.527}$
Victor Sreeram	$\frac{s+2.176}{s+0.84}$
State Feedback Controller	$\frac{s+12.7}{s+6.454}$
Observer Based Controller	$\frac{s+6.25}{s+0.0006454}$

3.3 Numerical Simulations

Example 1: Consider a following transfer function of a Chebyshev type 2 of 4^{th} order filter which is high pass with stop band, 11.001 dB ripple and stop band corner frequency 17 Hz.

$$G(s) = \frac{s^4 - 2.9e^{-15}s^3 + 289s^2 - 7.375e^{-13}s + 1.044e^4}{s^4 + 22.39s^3 + 539.6s^2 + 5826s + 3.704e^4}$$

with frequency interval $[\omega_1, \omega_2] = [7, 18]$ rad/sec.

Analysis & Discussion

Fig. 3.3 and 3.11 show the error $\sigma[G(s) - G_r(s)]$ plot, where $G_r(s)$ is 1^{st} and 2^{nd} LOAS obtained using existing and proposed techniques respectively. Fig. 3.4 and 3.12 represent

Table 3.2: LOAS for 2^{nd} order

Techniques	LOAS
Gawronski and Jaung	$\frac{s^2 - 0.1563s + 242.2}{s^2 - 4.531s + 348}$
Gugercin and Antaulos	$\frac{s^2 + 0.8167s + 268.7}{s^2 + 3.799s + 328.7}$
Victor Sreeram	$\frac{s^2 + 0.5517s + 257.4}{s^2 + 2.476s + 336.5}$
State Feedback Controller	$\frac{s^2 + 8.907s + 242.2}{s^2 + 4.531s + 348}$
Observer Based Controller	$\frac{s^2 + 4.376s - 105.8}{s^2 + 0.0004531s + 3.48e^{-6}}$

Table 3.3: Poles location of the LOAS

Techniques	1^{st} Order	2^{nd} Order
Gawronski and Jaung	6.454	$2.2657 \pm 18.517ij$
Gugercin and Antaulos	-2.5274	$-1.8994 \pm 18.031i$
Victor Sreeram	-0.83999	$-1.238 \pm 18.303ij$
State Feedback Controller	-6.454	$-2.2657 \pm 18.517i$
Observer Based Controller	-0.0006454	$-0.00022657 \pm 0.0018517i$

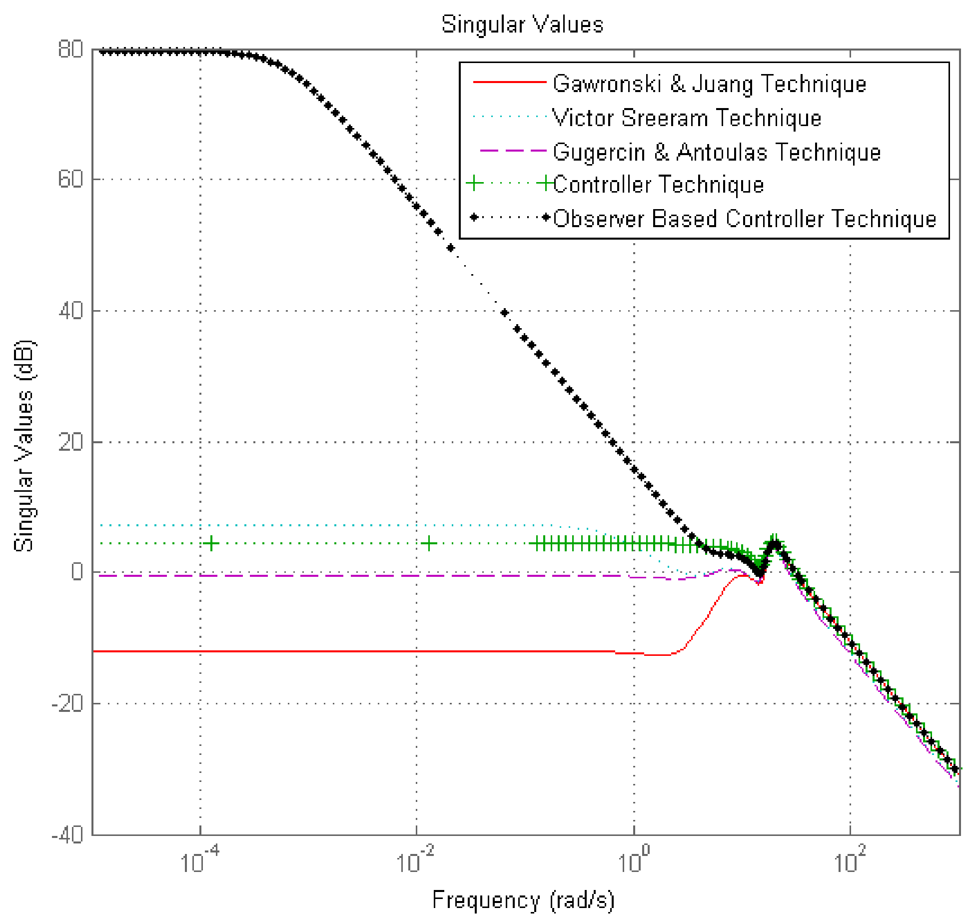

Figure 3.3: Error comparison for 1^{st} order LOAS

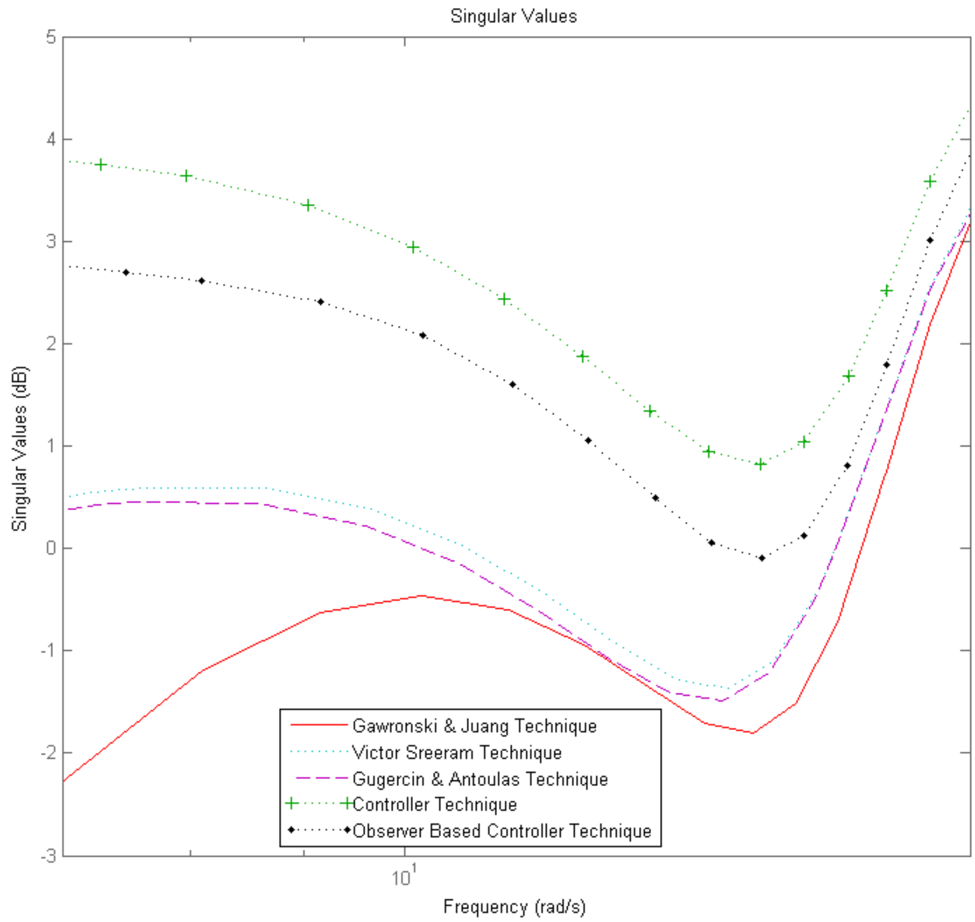

Figure 3.4: Error comparison - zoom-in view for 1^{st} order LOAS

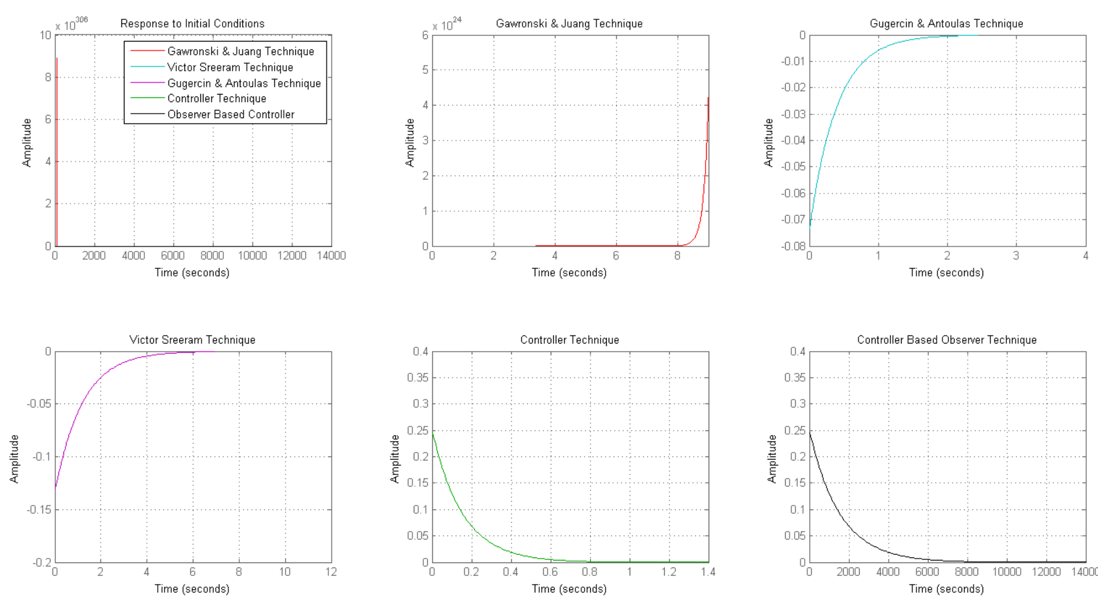

Figure 3.5: Natural response for 1^{st} order LOAS

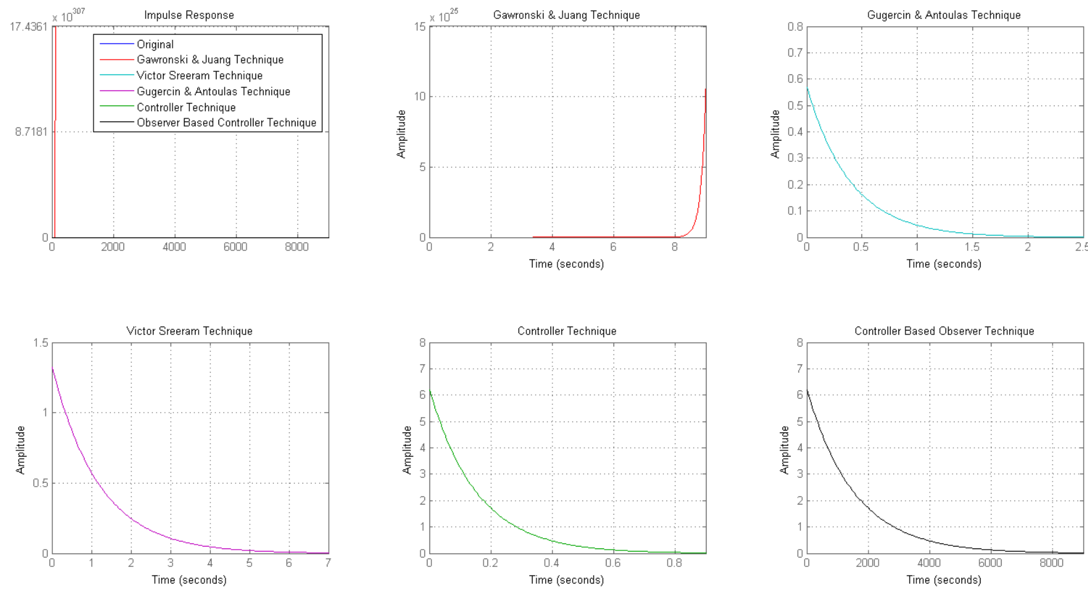

Figure 3.6: Impulse response for 1^{st} order LOAS

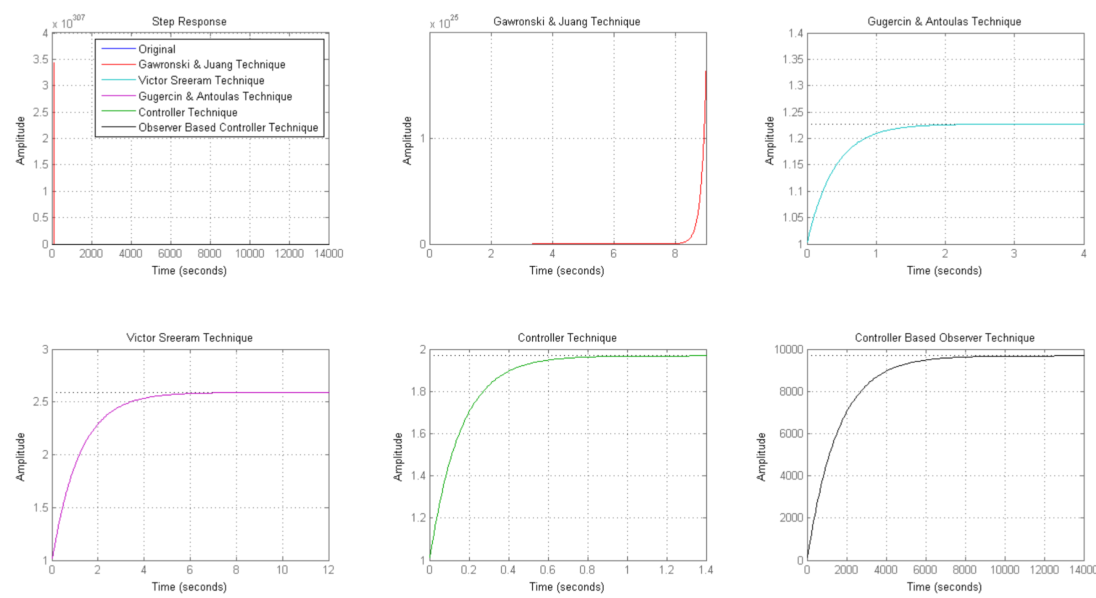

Figure 3.7: Step response for 1^{st} order LOAS

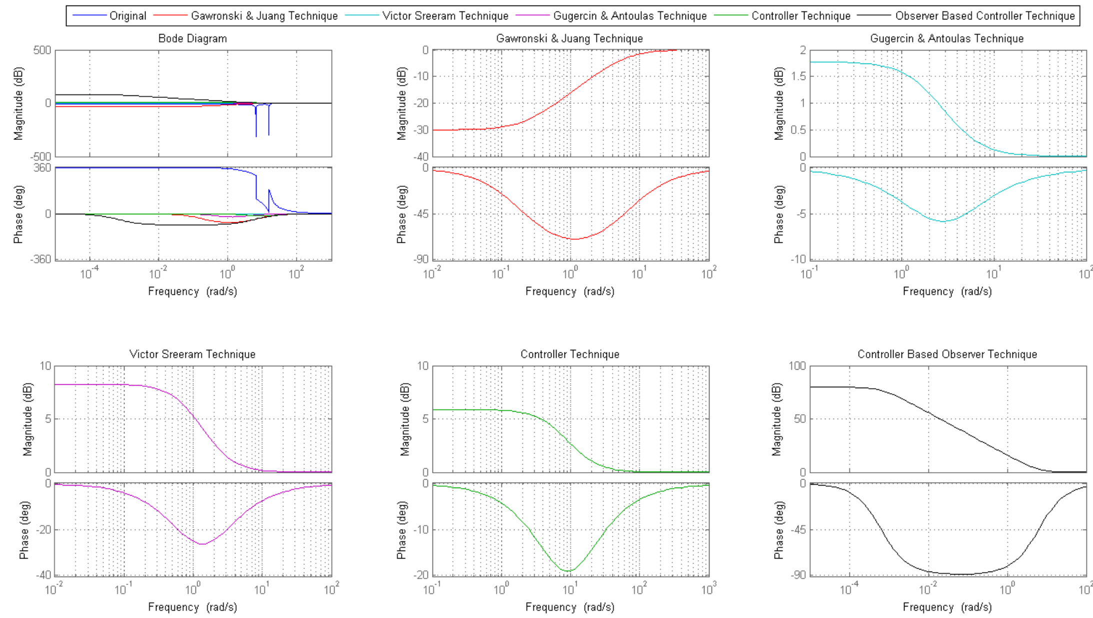

Figure 3.8: Bode plot for 1^{st} order LOAS

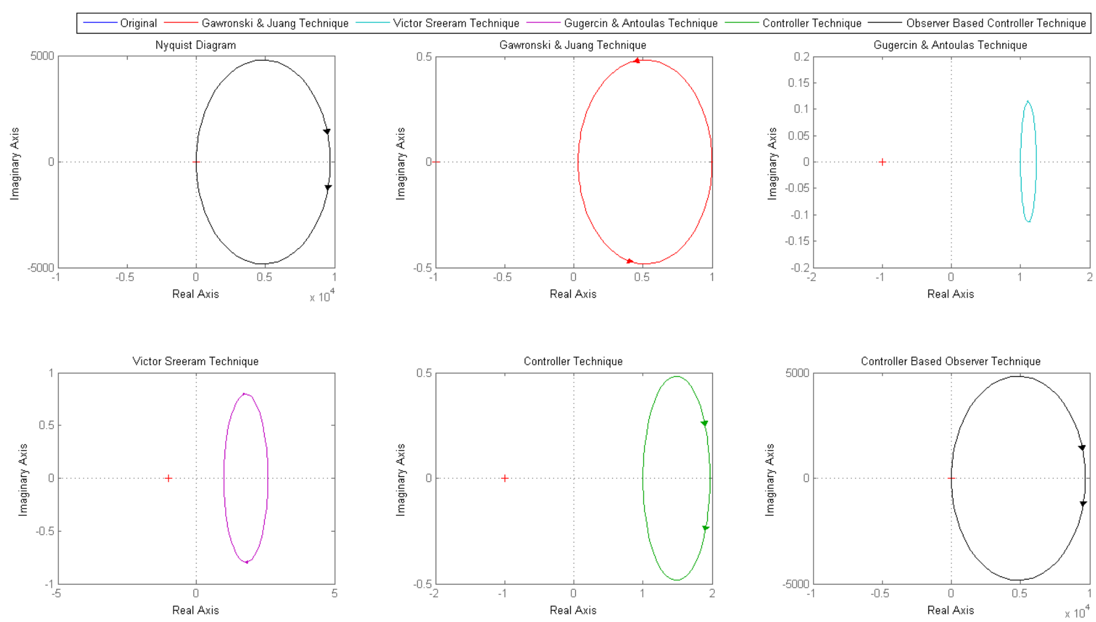

Figure 3.9: Nyquist plot for 1^{st} order LOAS

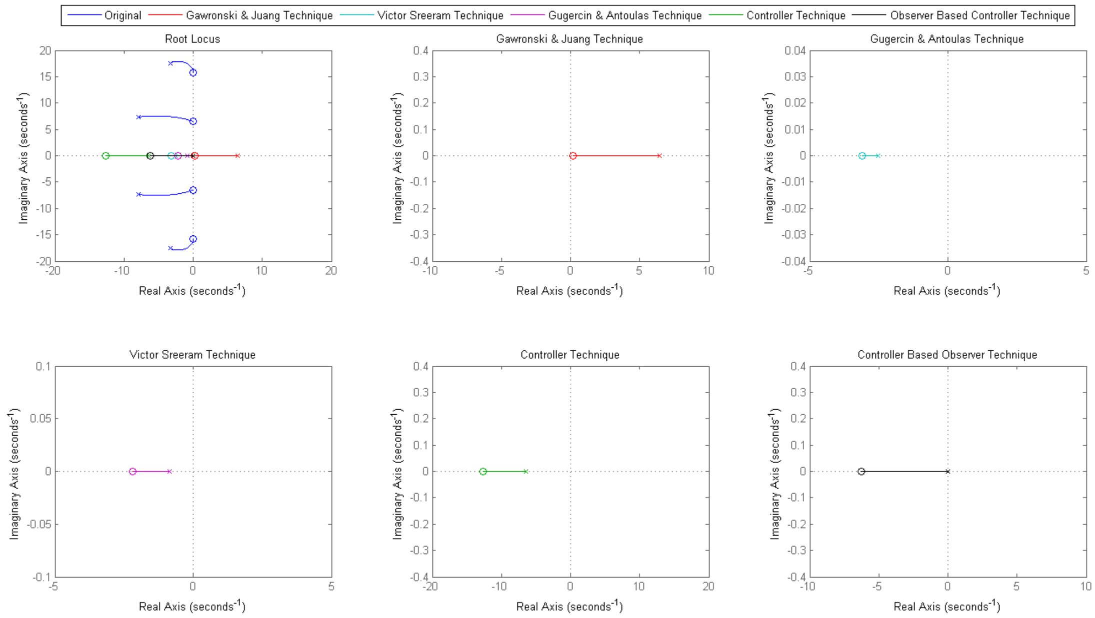

Figure 3.10: Root Locus plot for 1^{st} order LOAS

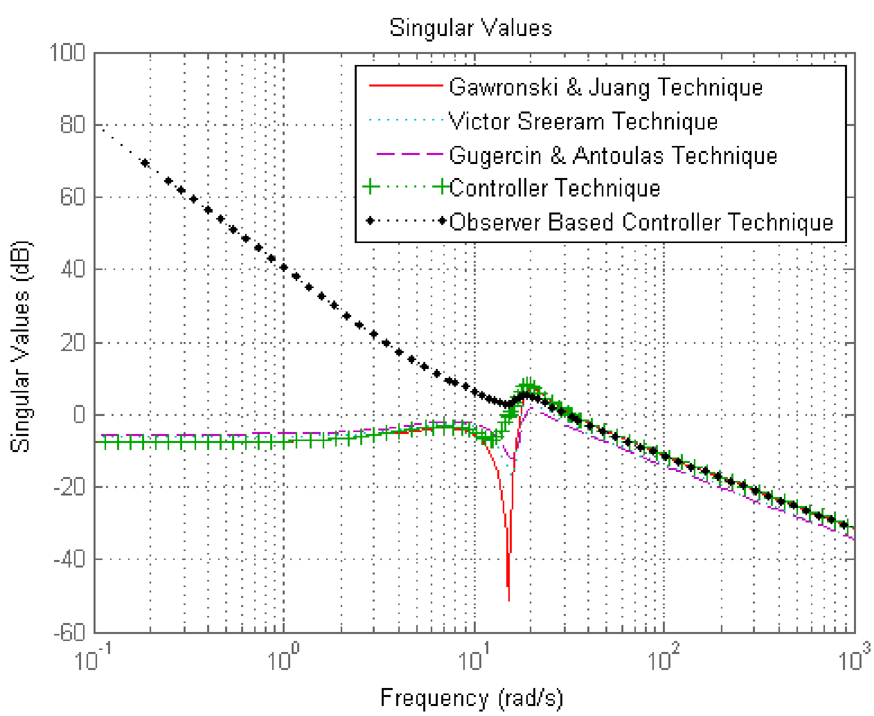

Figure 3.11: Error comparison for 2^{nd} order LOAS

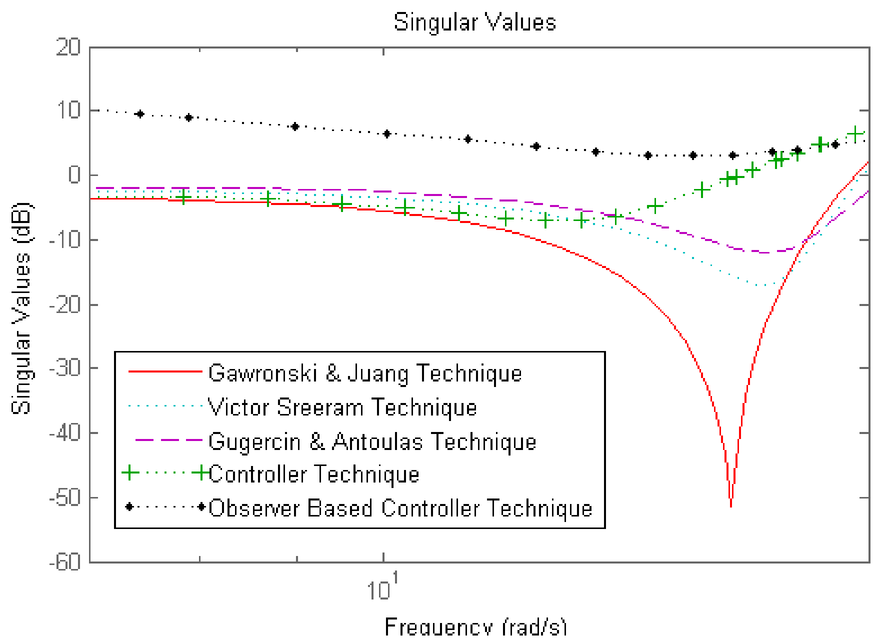

Figure 3.12: Error comparison - zoom-in view for 2^{nd} order LOAS

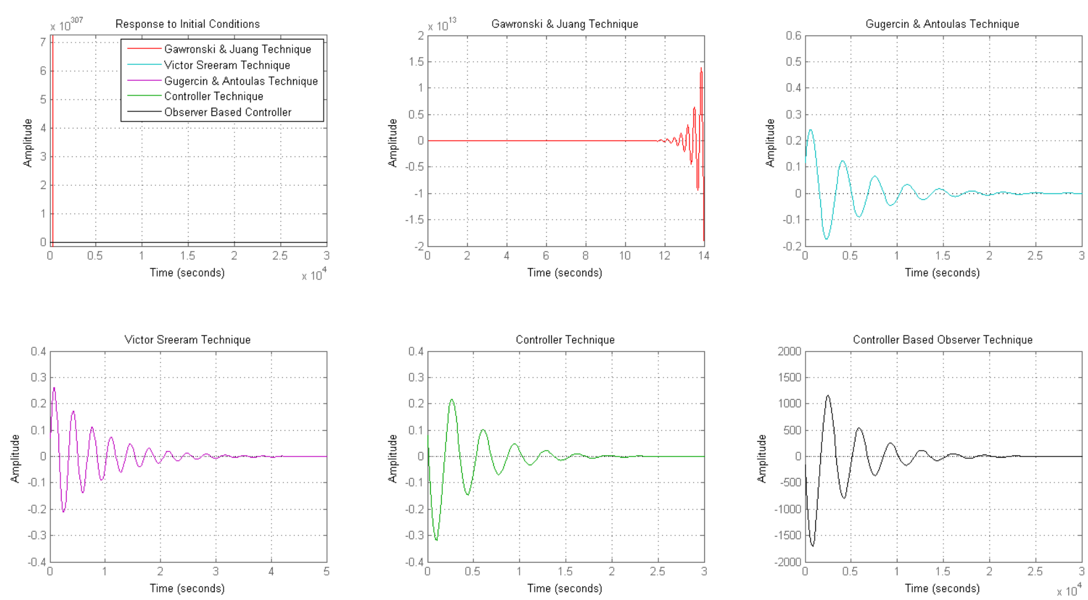

Figure 3.13: Natural response for 2^{nd} order LOAS

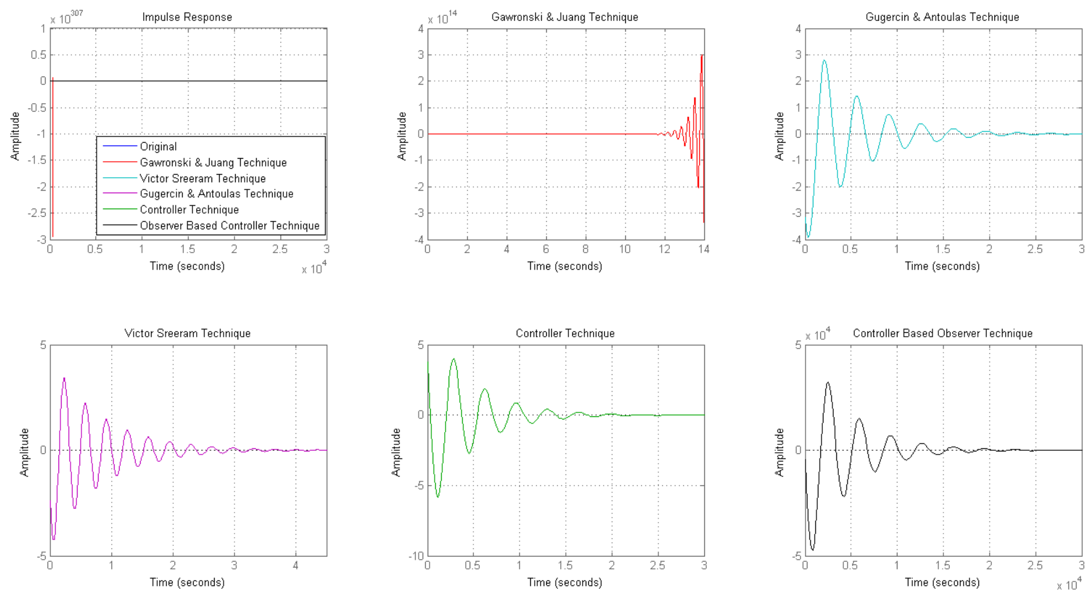

Figure 3.14: Impulse response for 2^{nd} order LOAS

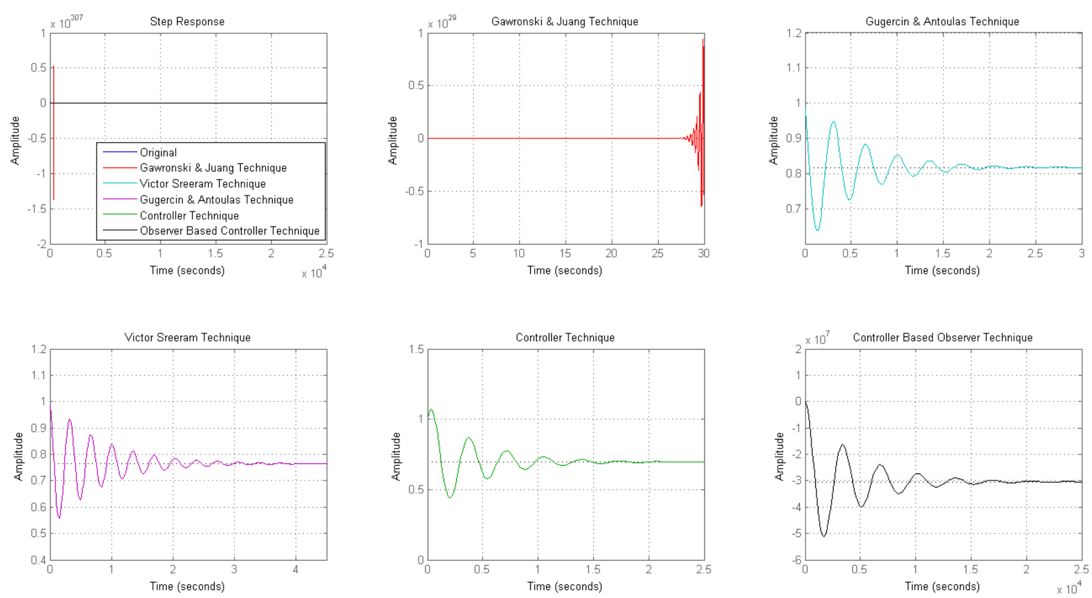

Figure 3.15: Step response for 2^{nd} order LOAS

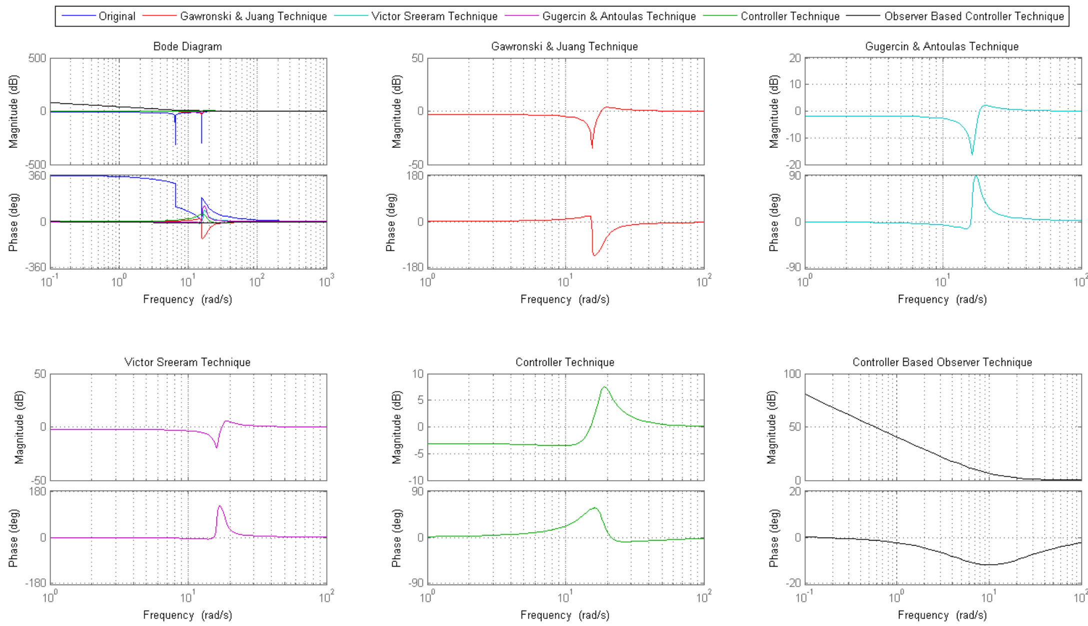

Figure 3.16: Bode plot for 2^{nd} order LOAS

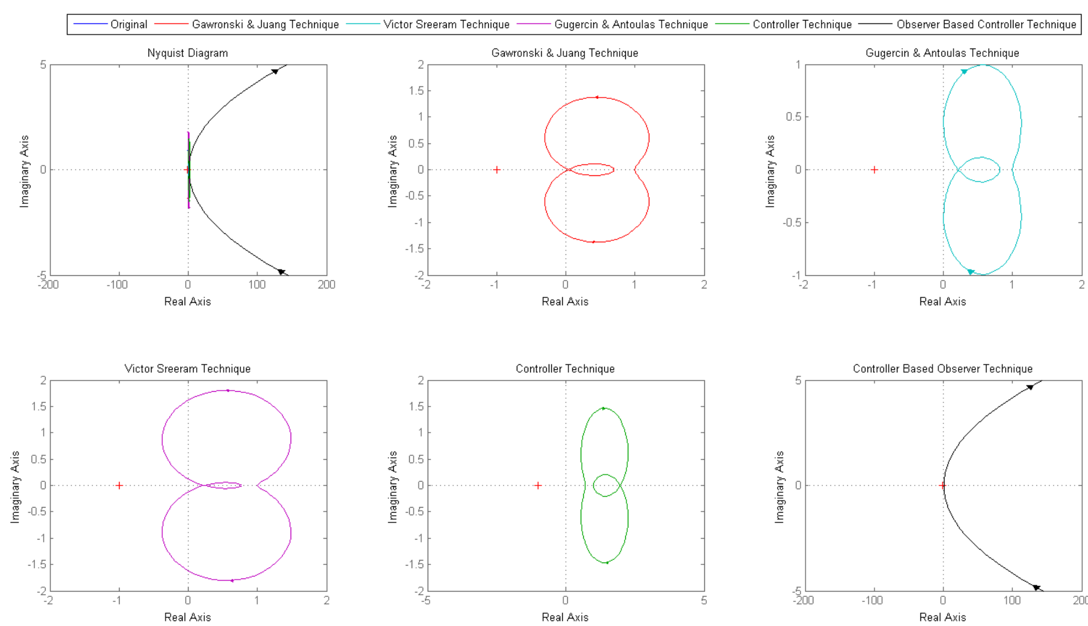

Figure 3.17: Nyquist plot for 2^{nd} order LOAS

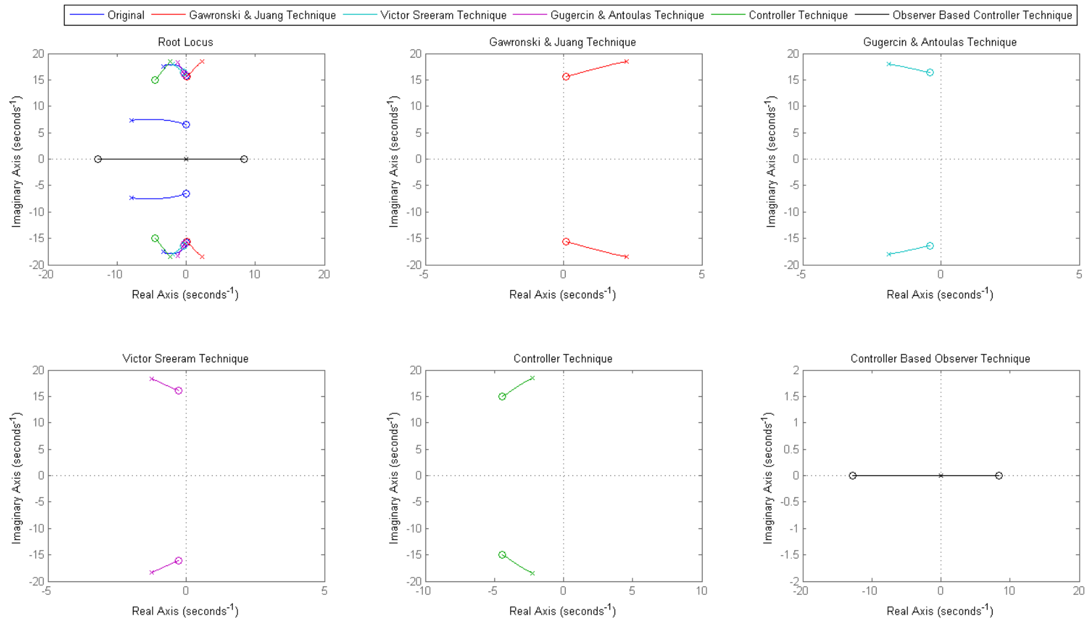

Figure 3.18: Root Locus plot for 2^{nd} order LOAS

zoom-in view of frequency response error. Whereas, Fig. (3.5, 3.13), (3.6, 3.14), (3.7, 3.15), (3.8, 3.16), (3.9, 3.17) and (3.10, 3.18) represents impulse response, natural response, bode Plot, step response, root locus plot and nyquist plot for the 1^{st} and 2^{nd} LOAS obtained using existing and proposed techniques respectively. It can be seen that the suggested technique gives low frequency response error. Simulation results show that GJ produces less error for approximation as regards other concurrent techniques, but it occasionally gives yields unstable LOAS as depicted in Table 3.1, 3.2 and 3.3. Proposed techniques produce less approximation errata (when analysed and compared with other techniques for preserving stability) and stable LOAS.

3.4 Conclusion

Novel frequency restrained LOAS method for continuous systems is suggested. The suggested method provide stability of LOAS and gives error bounds equation. The error for

LOAS is also improved and analysed with different current stability retaining frequency LOAS procedures. GJ procedure generates low error as related to other existing procedures, however, it often gives unstable LOAS.

Chapter 4

SIMULATION OF PRACTICAL APPLICATIONS

4.1 Simulation of Discrete Time Systems

Example:Consider a 4^{th} order mass spring system having following transfer function,

$$G(z) = \frac{z^4 + 1.881z^3 + 0.3322z^2 - 0.0277z + 0.0228}{z^4 - 0.265z^3 + 0.6974z^2 - 0.2011z + 0.2819}$$

with desired frequency interval $0.25\pi - 0.35\pi$.

Following LOAS obtains for each existing and proposed techniques,

For 1^{st} order LOAS:

$$G_{rwz}(z) = \frac{z + 2.082}{z - 0.04438}$$

$$G_{rvs}(z) = \frac{z + 2.145}{z - 0.0479}$$

$$G_{rb}(z) = \frac{z + 2.174}{z - 0.06168}$$

$$G_{rc}(z) = \frac{z + 2.082}{z - 0.04438}$$

$$G_{roc}(z) = \frac{z + 2.082}{z - 0.04438}$$

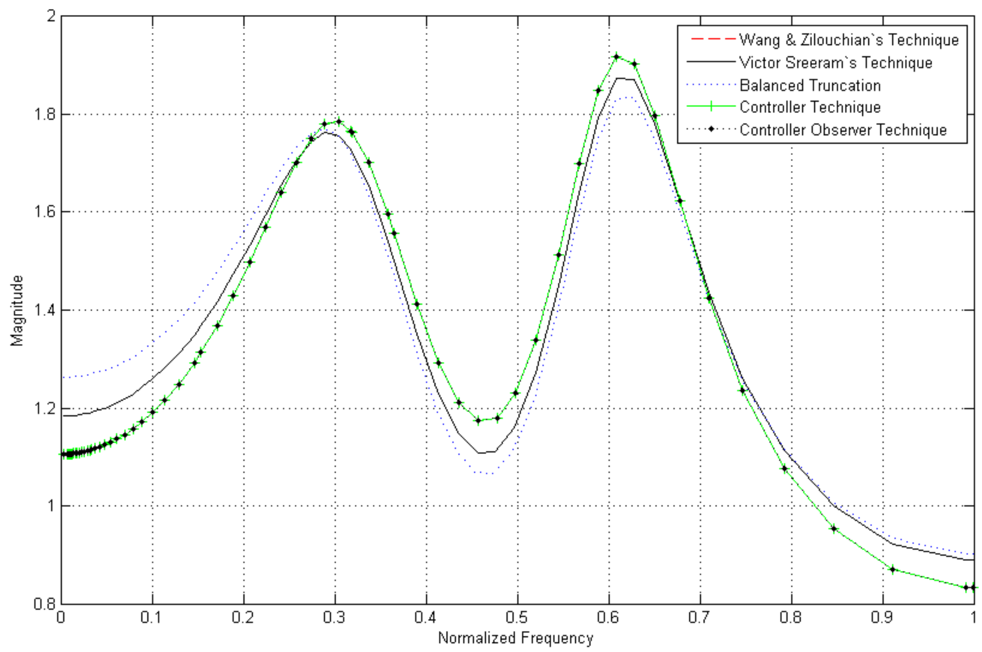

Figure 4.1: Error comparison for 1^{st} order LOAS

For 2^{nd} order LOAS:

$$G_{rwz}(z) = \frac{z^2 + 1.472z - 1.008}{z^2 - 0.6496z + 0.4088}$$

$$G_{rvs}(z) = \frac{z^2 + 1.568z - 1.012}{z^2 - 0.62z + 0.3943}$$

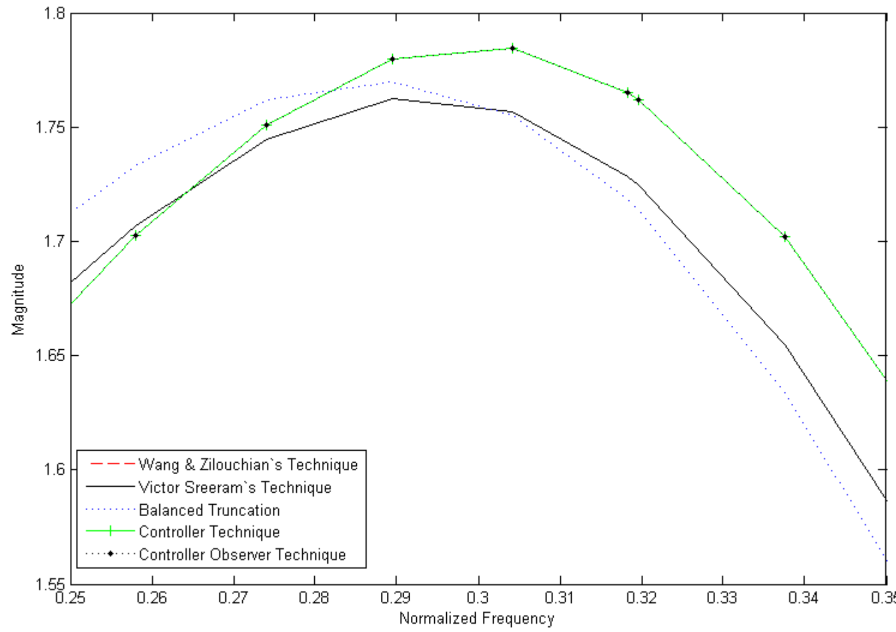

Figure 4.2: Error comparison - zoom-in view for 1^{st} order LOAS

$$G_{rb}(z) = \frac{z^2 + 2.06z + 0.216}{z^2 - 0.1756z + 0.4417}$$

$$G_{rc}(z) = \frac{z^2 + 1.472z - 1.008}{z^2 - 0.6496z + 0.4088}$$

$$G_{roc}(z) = \frac{z^2 + 1.472z - 1.008}{z^2 - 0.6496z + 0.4088}$$

For 3^{rd} order LOAS:

$$G_{rwz}(z) = \frac{z^3 + 1.52z^2 - 1.415z + 0.3989}{z^3 - 0.7479z^2 + 0.4639z - 0.006836}$$

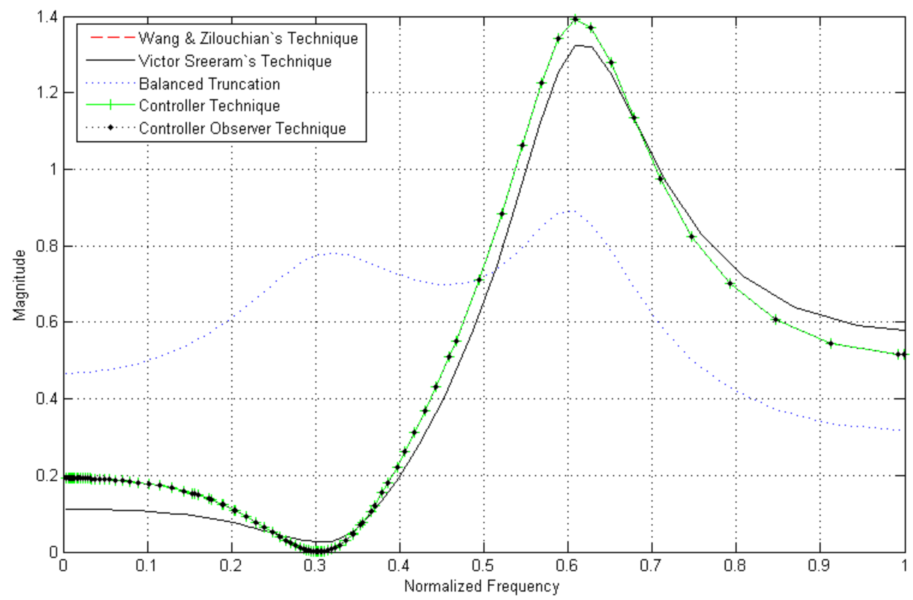

Figure 4.3: Error comparison for 2^{nd} order LOAS

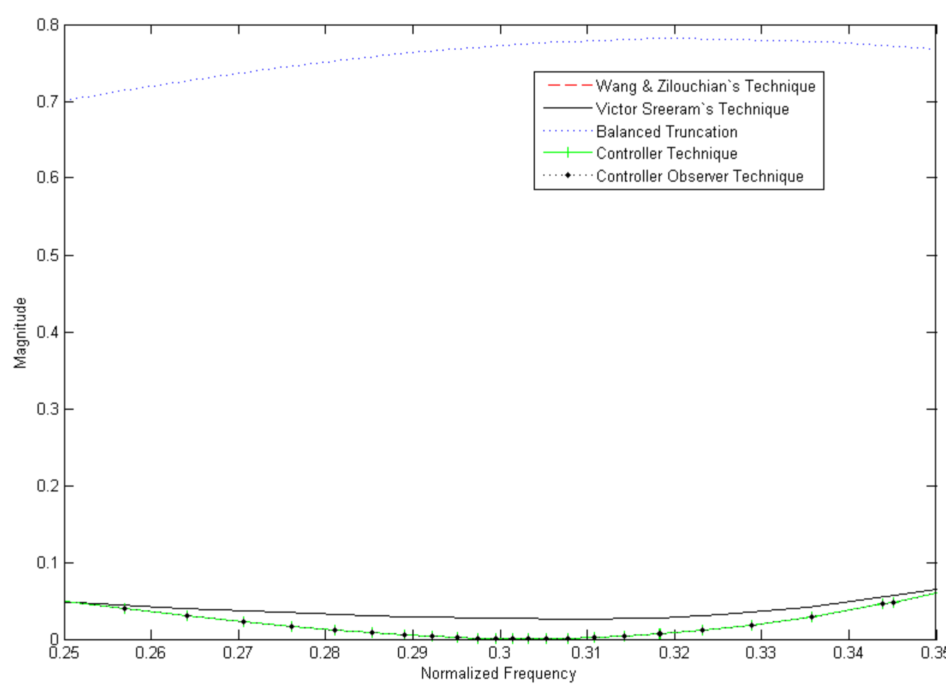

Figure 4.4: Error comparison - zoom-in view for 2^{nd} order LOAS

$$G_{rvs}(z) = \frac{z^3 + 1.824z^2 - 0.2002z - 0.804}{z^3 - 0.2359z^2 + 0.1372z + 0.103}$$

$$G_{rb}(z) = \frac{z^3 + 1.883z^2 - 0.1498z - 0.946}{z^3 - 0.2623z^2 + 0.2062z - 0.05451}$$

$$G_{rc}(z) = \frac{z^3 + 1.52z^2 - 1.415z + 0.3989}{z^3 - 0.7479z^2 + 0.4639z - 0.006836}$$

$$G_{roc}(z) = \frac{z^3 + 1.52z^2 - 1.415z + 0.3989}{z^3 - 0.7479z^2 + 0.4639z - 0.006836}$$

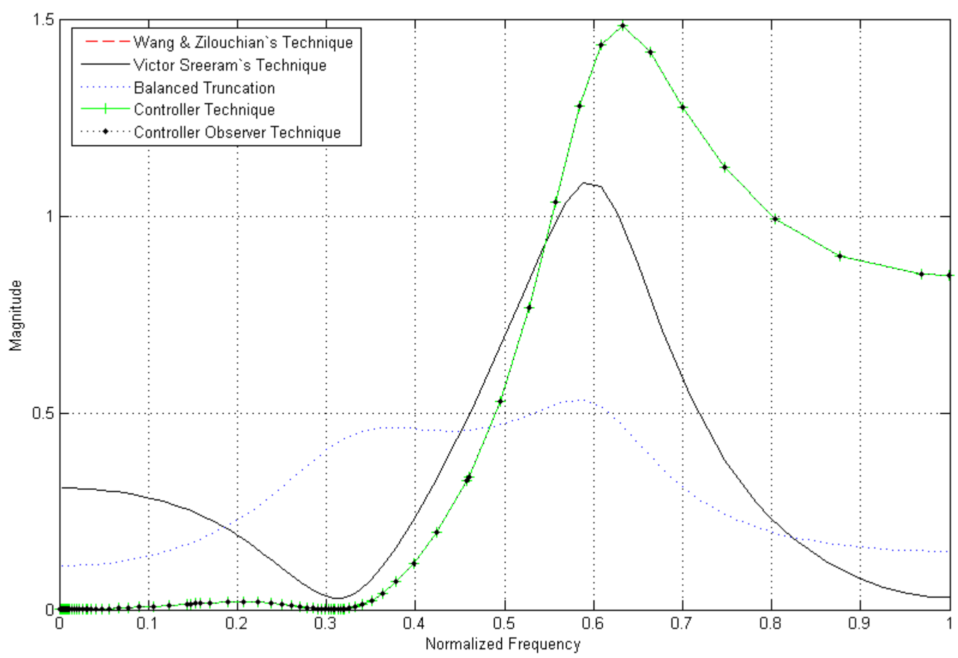

Figure 4.5: Error comparison for 3^{rd} order LOAS

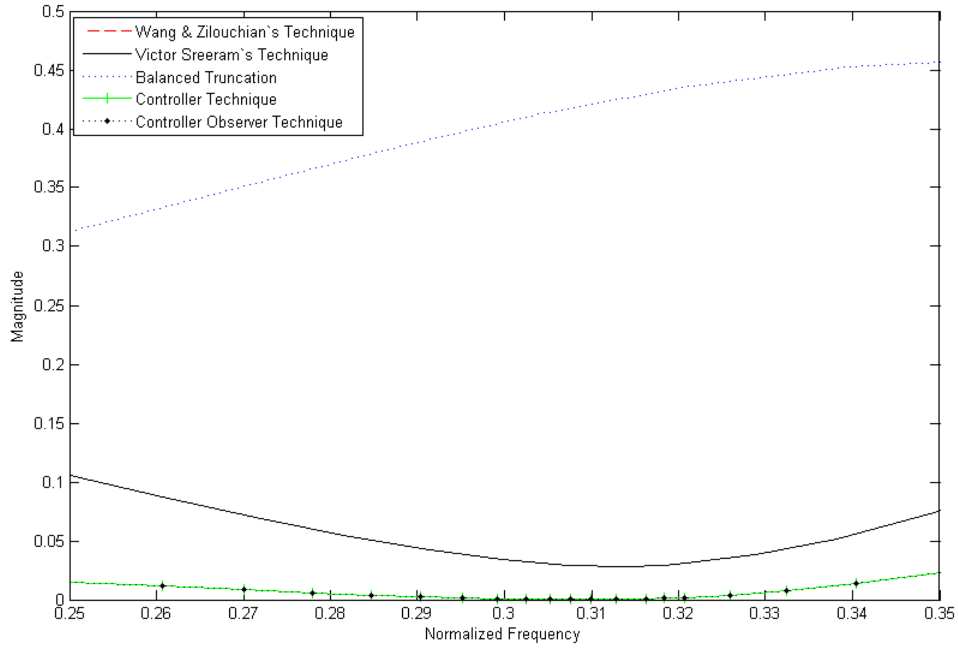

Figure 4.6: Error comparison - zoom-in view for 3^{rd} order LOAS

4.2 Simulation of Continuous Time Systems

Example 1:Consider a 6^{th} order quarter car model having following transfer function,

$$G(s) = \frac{-2.1s^4 - .24s^3 - 24.8s^2 - .9s - 45.3}{s^6 + 0.3s^5 + 32.9s^4 + 3.6s^3 + 180s^2 + 3.5s + 119}$$

with desired frequency interval $[\omega_1, \omega_2] = [10, 15]$ rad/sec.

Following LOAS obtains for each existing and proposed techniques,

For 1^{st} order LOAS:

$$G_{rgj}(s) = \frac{-0.08911}{s - 4.826}$$

$$G_{rvs}(s) = \frac{-0.1236}{s + 0.0001052}$$

$$G_{rga}(s) = \frac{-0.119}{s + 0.003474}$$

$$G_{rc}(s) = \frac{-0.08911}{s + 4.826}$$

$$G_{roc}(s) = \frac{-0.08911}{s + 0.0004826}$$

For 2^{nd} order LOAS:

$$G_{rgj}(s) = \frac{0.0002342s - 2.106}{s^2 + 0.2381s + 22.58}$$

$$G_{rvs}(s) = \frac{0.00344s - 0.2208}{s^2 + 0.01054s + 0.7664}$$

$$G_{rga}(s) = \frac{0.002426s - 0.2099}{s^2 + 0.007725s + 0.7623}$$

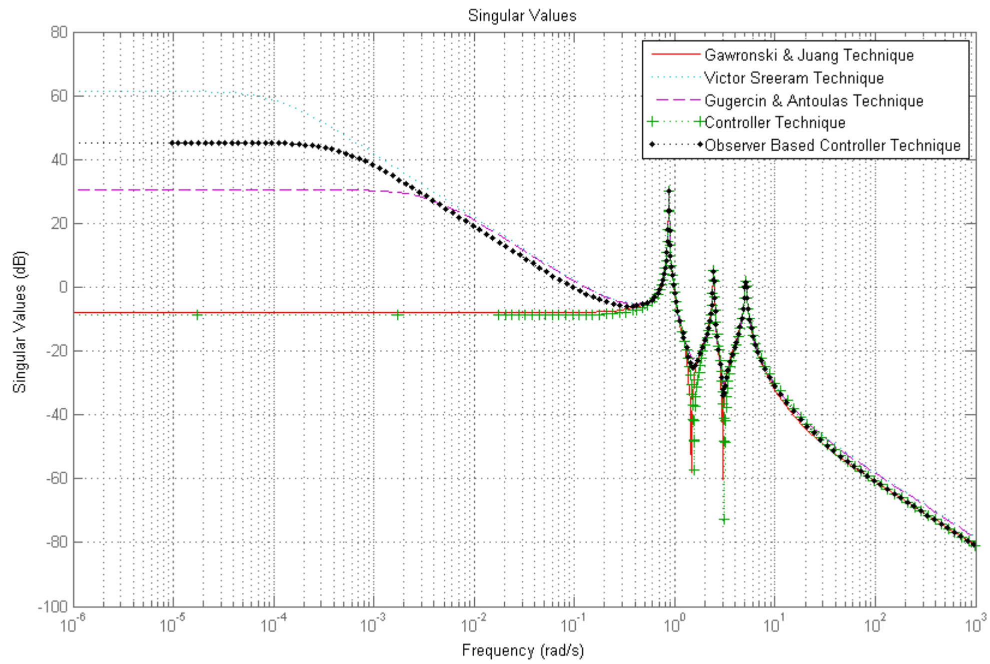

Figure 4.7: Error comparison for 1^{st} order LOAS

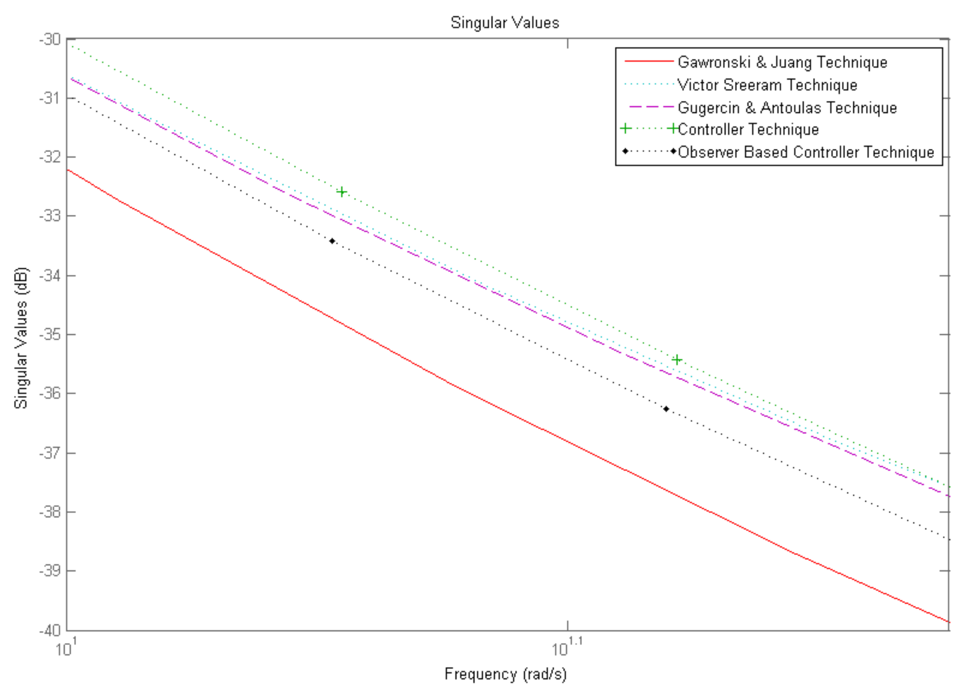

Figure 4.8: Error comparison - zoom-in view for 1^{st} order LOAS

$$G_{rc}(s) = \frac{0.0002342s - 2.106}{s^2 + 0.2381s + 22.58}$$

$$G_{roc}(s) = \frac{0.0002342s - 2.106}{s^2 + 0.2381s + 22.58}$$

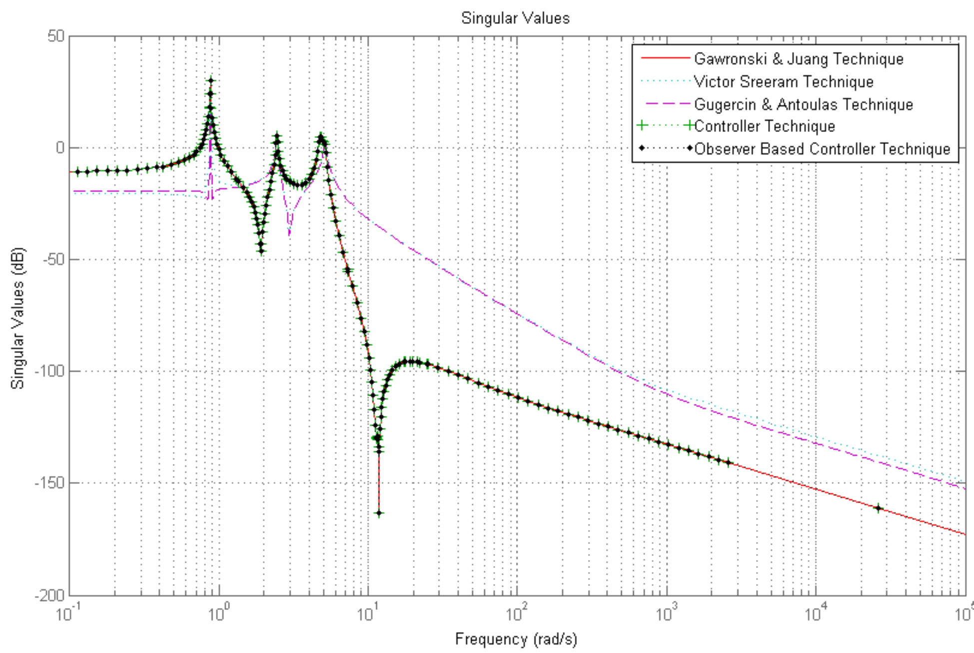

Figure 4.9: Error comparison for 2^{nd} order LOAS

For 3^{rd} order LOAS:

$$G_{rgj}(s) = \frac{-0.0003526s^2 - 2.108s + 11.75}{s^3 - 5.412s^2 + 21.22s - 131.4}$$

$$G_{rvs}(s) = \frac{0.1586s^2 - 0.2392s + 0.08206}{s^3 + 0.1934s^2 + 0.7674s + 0.1378}$$

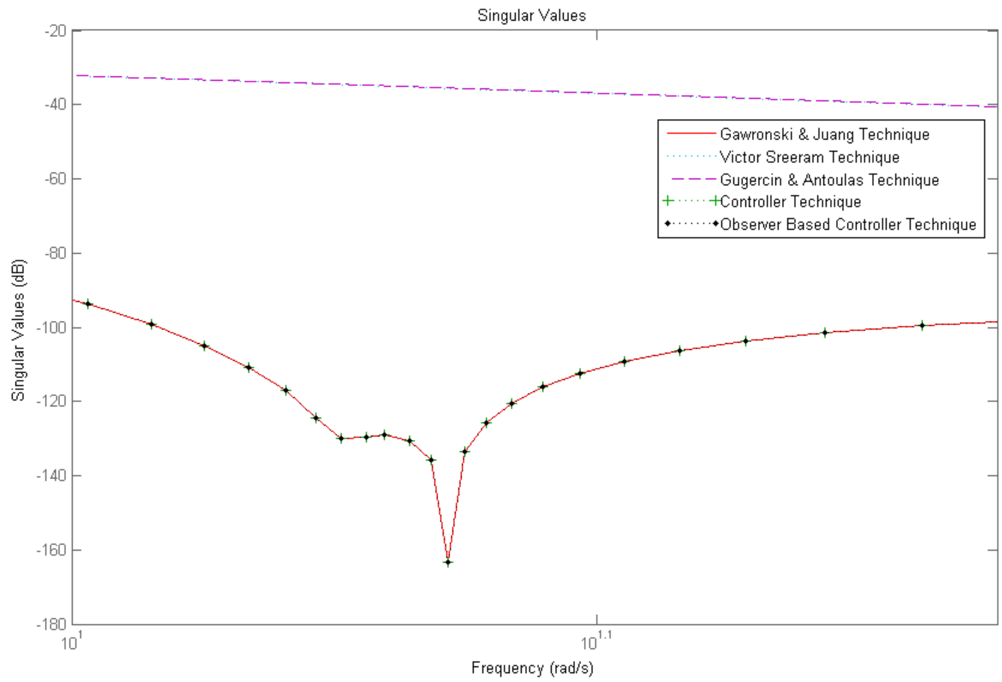

Figure 4.10: Error comparison - zoom-in view for 2^{nd} order LOAS

$$G_{rga}(s) = \frac{0.1664s^2 - 0.1975s + 0.1151}{s^3 + 0.1446s^2 + 0.7663s + 0.1062}$$

$$G_{rc}(s) = \frac{-0.0003526s^2 - 2.108s + 11.75}{s^3 + 6.026s^2 + 24.73s + 131.4}$$

$$G_{roc}(s) = \frac{-0.0003526s^2 - 2.108s + 11.75}{s^3 + 0.3078s^2 + 22.97s + 0.01314}$$

For 4^{th} order LOAS:

$$G_{rgj}(s) = \frac{1.801e^{-7}s^3 - 2.118s^2 - 0.179s - 17.41}{s^4 + 0.2969s^3 + 29.46s^2 + 1.755s + 85.03}$$

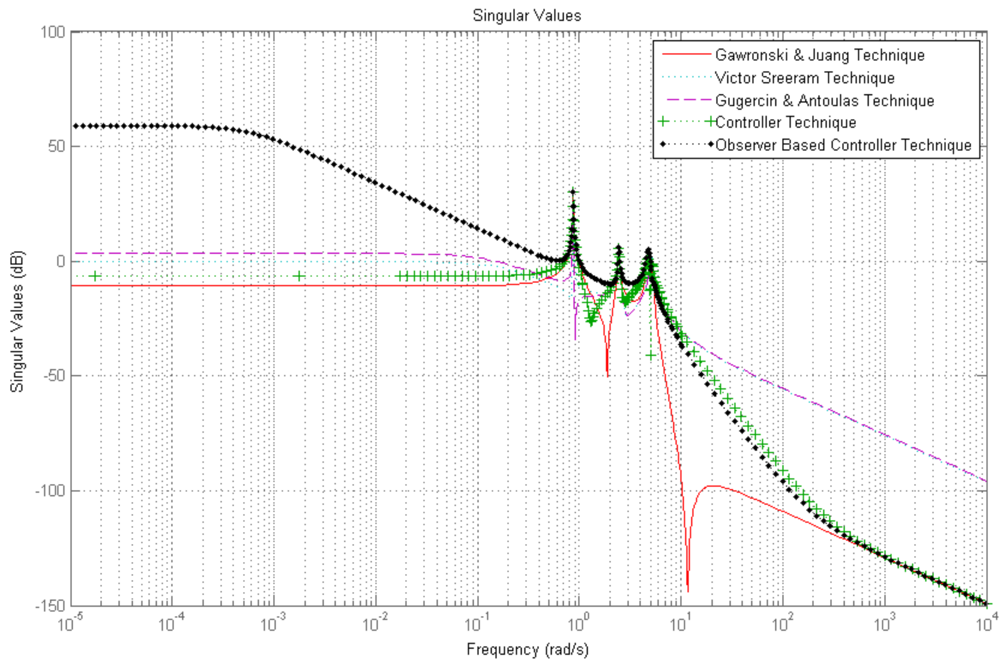

Figure 4.11: Error comparison for 3^{rd} order LOAS

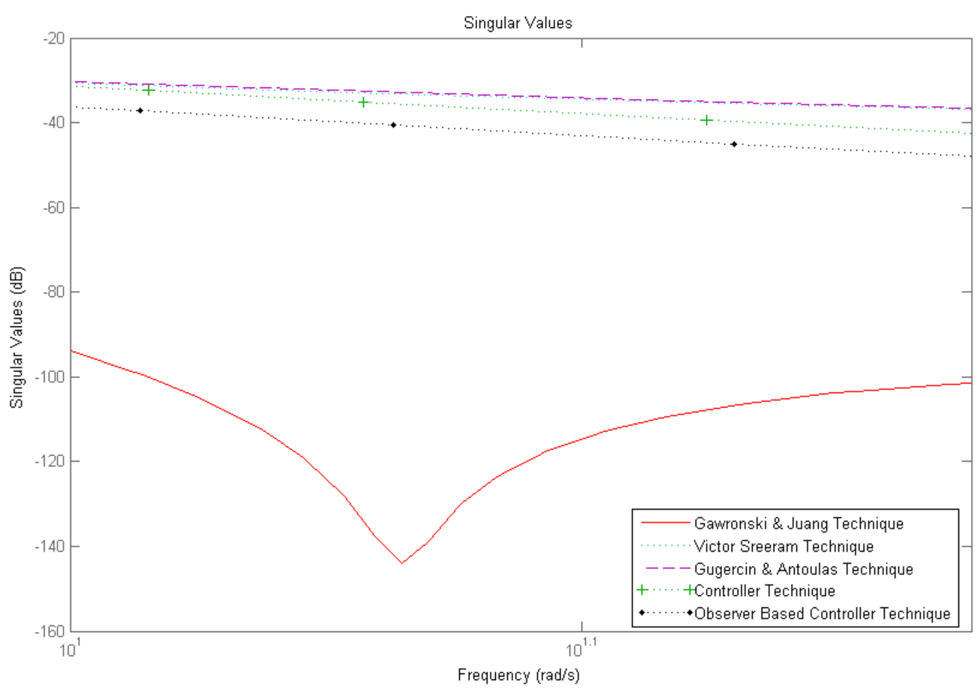

Figure 4.12: Error comparison - zoom-in view for 3^{rd} order LOAS

$$G_{rvs}(s) = \frac{0.0087s^3 - 1.866s^2 - 0.09502s - 6.771}{s^4 + 0.2733s^3 + 27.02s^2 + 0.4052s + 20.04}$$

$$G_{rga}(s) = \frac{0.009328s^3 - 1.857s^2 - 0.09112s - 6.755}{s^4 + 0.2705s^3 + 27.02s^2 + 0.4001s + 20.04}$$

$$G_{rc}(s) = \frac{1.801e^{-7}s^3 - 2.118s^2 - 0.179s - 17.41}{s^4 + 0.2969s^3 + 29.46s^2 + 1.755s + 85.03}$$

$$G_{roc}(s) = \frac{1.801e^{-7}s^3 - 2.118s^2 - 0.179s - 17.41}{s^4 + 0.2969s^3 + 29.46s^2 + 1.755s + 85.03}$$

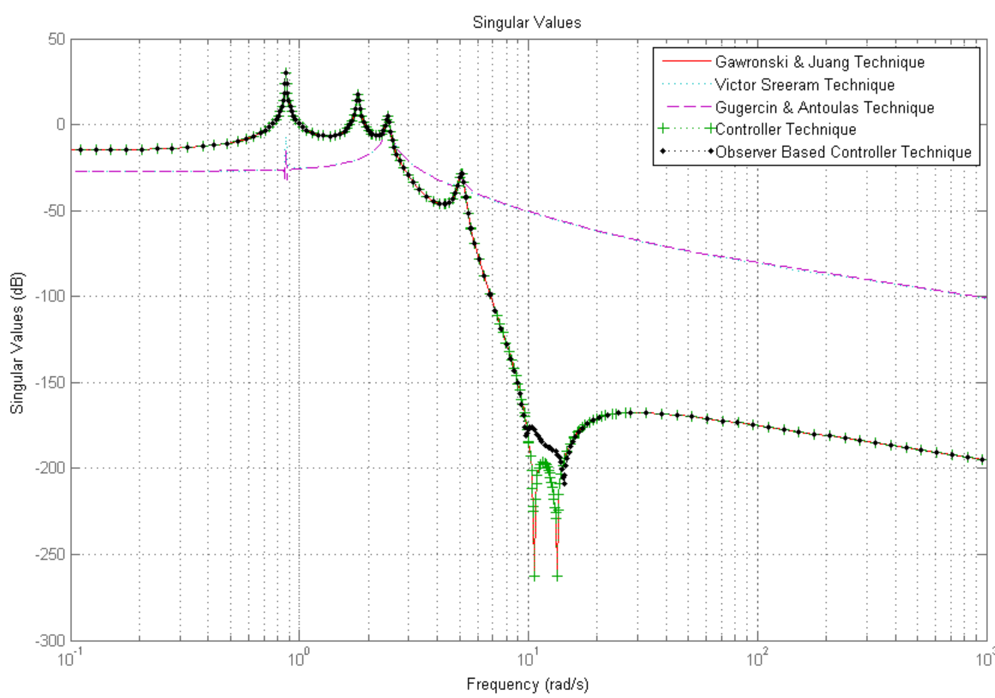

Figure 4.13: Error comparison for 4^{th} order LOAS

For 5^{th} order LOAS:

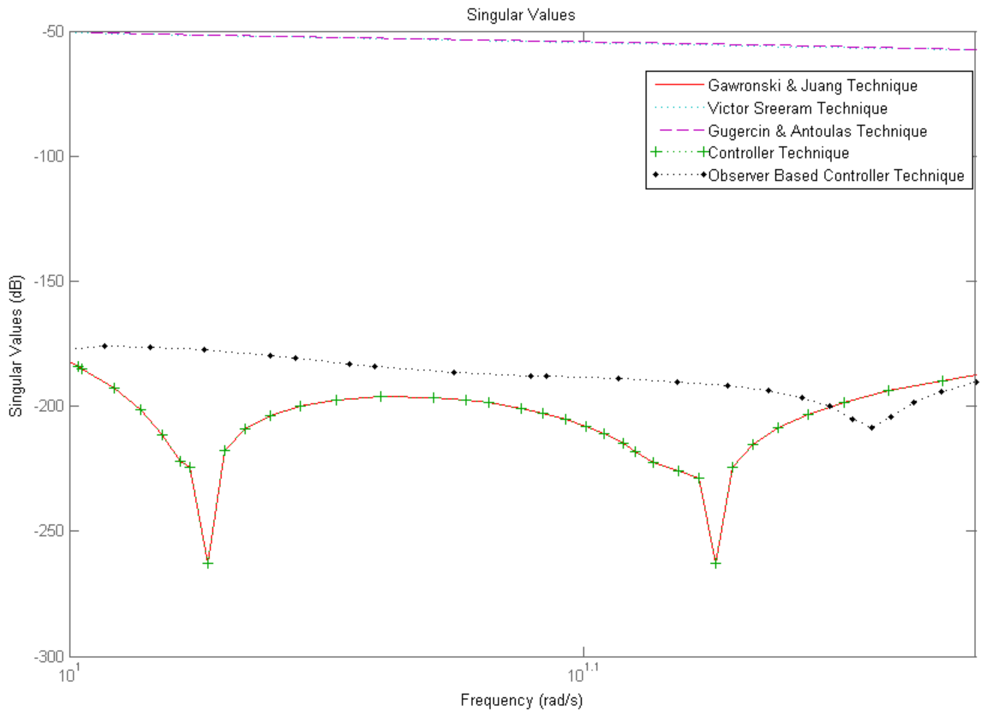

Figure 4.14: Error comparison - zoom-in view for 4^{th} order LOAS

$$G_{rgj}(s) = \frac{-1.07e^{-7}s^4 - 2.118s^3 + 11.93s^2 - 16.4s + 100.6}{s^5 - 5.421s^4 + 27.77s^3 - 167.2s^2 + 75.06s - 497.9}$$

$$G_{rvs}(s) = \frac{0.05395s^4 - 1.854s^3 + 1.189s^2 - 6.833s + 0.7061}{s^5 + 0.3243s^4 + 27.03s^3 + 1.864s^2 + 20.06s + 1.081}$$

$$G_{rga}(s) = \frac{0.05784s^4 - 1.84s^3 + 1.283s^2 - 6.755s + 0.8813}{s^5 + 0.3016s^4 + 27.03s^3 + 1.285s^2 + 20.06s + 0.658}$$

$$G_{rc}(s) = \frac{-1.07e^{-7}s^4 - 2.118s^3 + 11.93s^2 - 16.4s + 100.6}{s^5 + 6.039s^4 + 31.31s^3 + 171.4s^2 + 98.73s + 497.9}$$

$$G_{roc}(s) = \frac{-1.07e^{-7}s^4 - 2.118s^3 + 11.93s^2 - 16.4s + 100.6}{s^5 + 0.3093s^4 + 29.54s^3 + 1.851s^2 + 178.8s - 425}$$

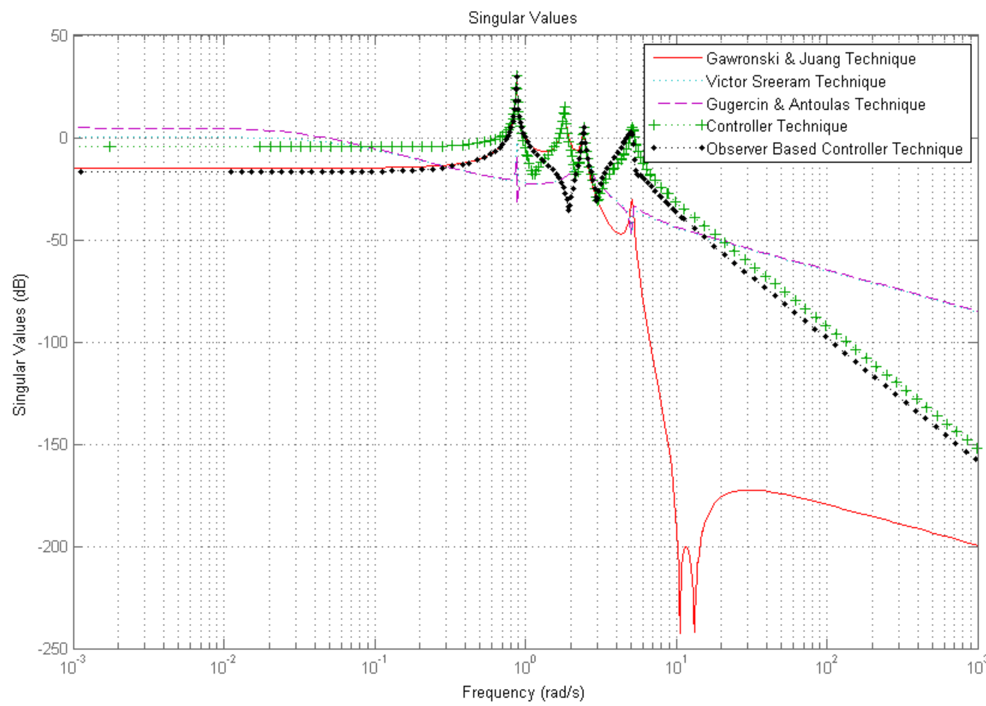

Figure 4.15: Error comparison for 5^{th} order LOAS

Example 2: Consider a 16^{th} order chebyshev 1 type filter which passes frequencies between $(10-17)\pi$ and 15 db of ripples in pass band having following transfer function

$$G(s) = \frac{\begin{array}{r} 8139s^8 - 3.76e^{-10}s^7 - 7.096e - 10s^6 - 1.012e^{-8}s^5 + 1.583e^{-6}s^4 + 1.558e^{-6}s^3 \\ +0.0001065s^2 + 0.0002075s - 0.001052 \end{array}}{\begin{array}{r} s^{16} + 0.8062s^{15} + 1458s^{14} + 1025s^{13} + 9.125e^5s^{12} + 5.467e^5s^{11} \\ +3.198e^8s^{10} + 1.584e^8s^9 + 6.866e^{10}s^8 + 2.693e^{10}s^7 + 9.243e^{12}s^6 + 2.686e^{12}s^5 \\ +7.621e^{14}s^4 + 1.456e^{14}s^3 + 3.52e^{16}s^2 + 3.308e^{15}s + 6.976e^{17} \end{array}}$$

Following LOAS obtains for each existing and proposed techniques,

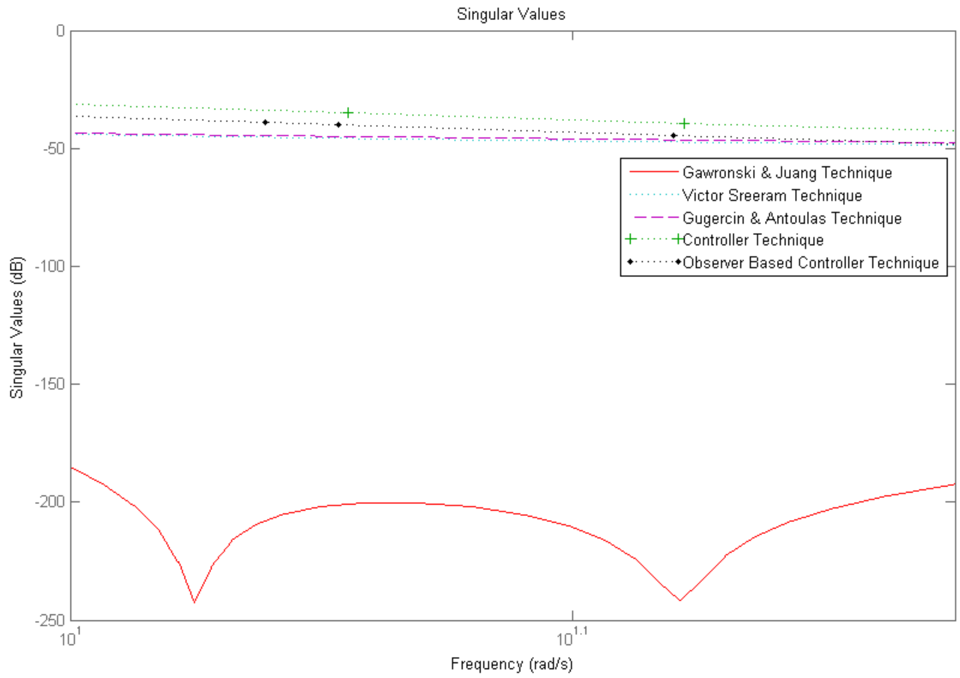

Figure 4.16: Error comparison - zoom-in view for 5^{th} order LOAS

For 1^{st} order LOAS:

$$G_{rgj}(s) = \frac{0.003444}{s + 0.1387}$$

$$G_{rvs}(s) = \frac{-0.002426}{s + 0.005066}$$

$$G_{rga}(s) = \frac{-0.01705}{s + 0.01371}$$

$$G_{rc}(s) = \frac{0.003444}{s + 0.1387}$$

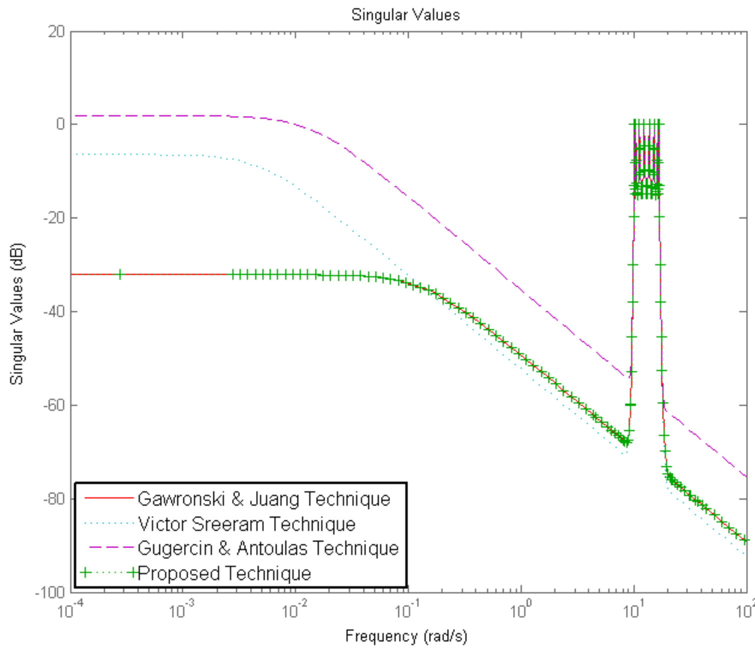

Figure 4.17: Error comparison in entire frequency range for 1^{st} order LOAS

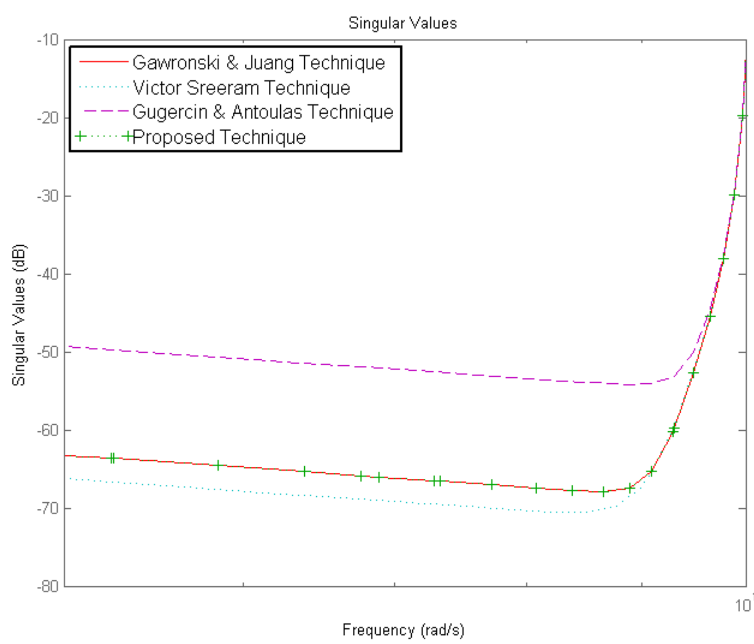

Figure 4.18: Error comparison plot in pin point frequency interval for 1^{st} order LOAS

For 2^{nd} order LOAS:

$$G_{rgj}(s) = \frac{-0.0017s + 0.08309}{s^2 + 0.009129s + 100.1}$$

$$G_{rvs}(s) = \frac{-0.02136s - 0.04411}{s^2 + 0.0203s + 102.4}$$

$$G_{rga}(s) = \frac{-0.02529s + 0.009708}{s^2 + 0.01801s + 102.1}$$

$$G_{rc}(s) = \frac{-0.0017s + 0.08309}{s^2 + 0.009129s + 100.1}$$

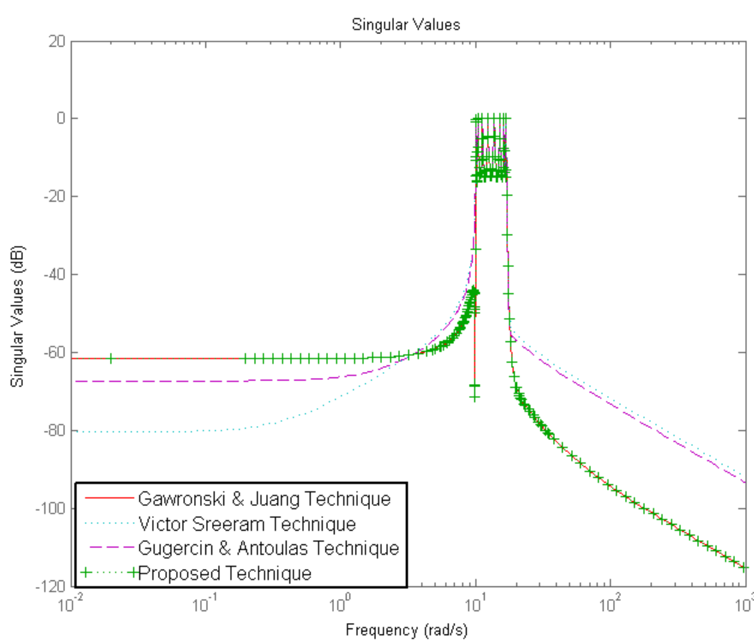

Figure 4.19: Error comparison in entire frequency range for 2^{nd} order LOAS

For 3^{rd} order LOAS:

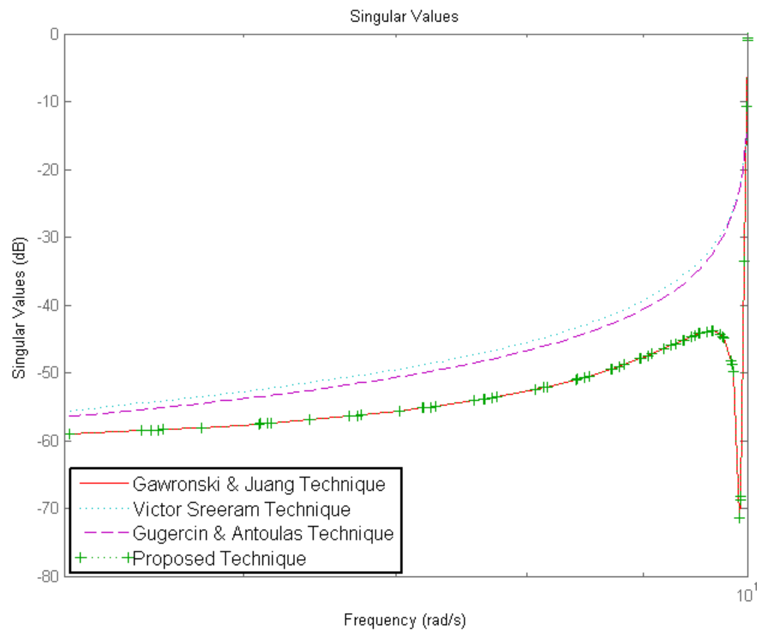

Figure 4.20: Error comparison plot in pin point frequency interval for 2^{nd} order LOAS

$$G_{rgj}(s) = \frac{-0.006734s^2 + 0.08279s - 0.3869}{s^3 + 1.076s^2 + 100.2s + 107}$$

$$G_{rvs}(s) = \frac{-0.01314s^2 - 0.04678s + 0.8388}{s^3 + 0.02879s^2 + 102.4s + 0.8765}$$

$$G_{rga}(s) = \frac{-0.009374s^2 + 0.002604s + 1.617}{s^3 + 0.04225s^2 + 102.1s + 2.514}$$

$$G_{rc}(s) = \frac{-0.006734s^2 + 0.08279s - 0.3869}{s^3 + 1.076s^2 + 100.2s + 107}$$

For 4^{th} order LOAS:

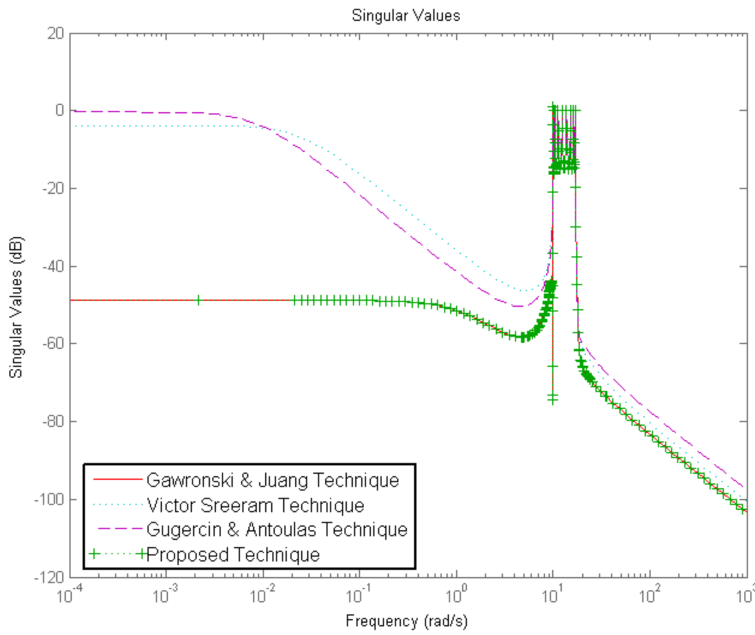

Figure 4.21: Error comparison in entire frequency range for 3^{rd} order LOAS

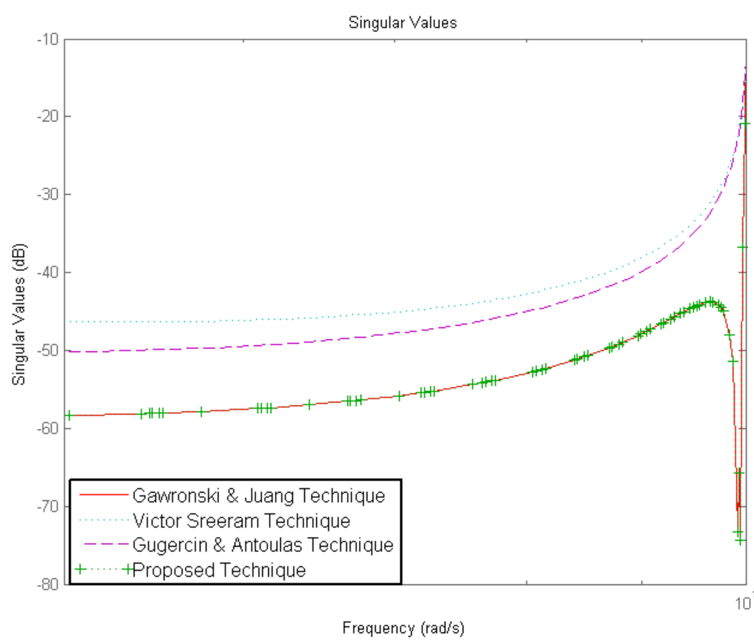

Figure 4.22: Error comparison plot in pin point frequency interval for 3^{rd} order LOAS

$$G_{rgj}(s) = \frac{0.0003919s^3 - 0.02315s^2 + 0.03128s - 1.785}{s^4 + 0.02109s^3 + 203s^2 + 2.188s + 1.03e^4}$$

$$G_{rvs}(s) = \frac{-0.03895s^3 + 0.5305s^2 - 4.601s + 54.79}{s^4 + 0.05388s^3 + 226.2s^2 + 5.948s + 1.267e^4}$$

$$G_{rga}(s) = \frac{-0.04579s^3 + 0.3244s^2 - 5.316s + 32.48}{s^4 + 0.05703s^3 + 229.7s^2 + 6.421s + 1.304e^4}$$

$$G_{rc}(s) = \frac{0.0003919s^3 - 0.02315s^2 + 0.03128s - 1.785}{s^4 + 0.2s^3 + 203s^2 + 20.3s + 1.03e^4}$$

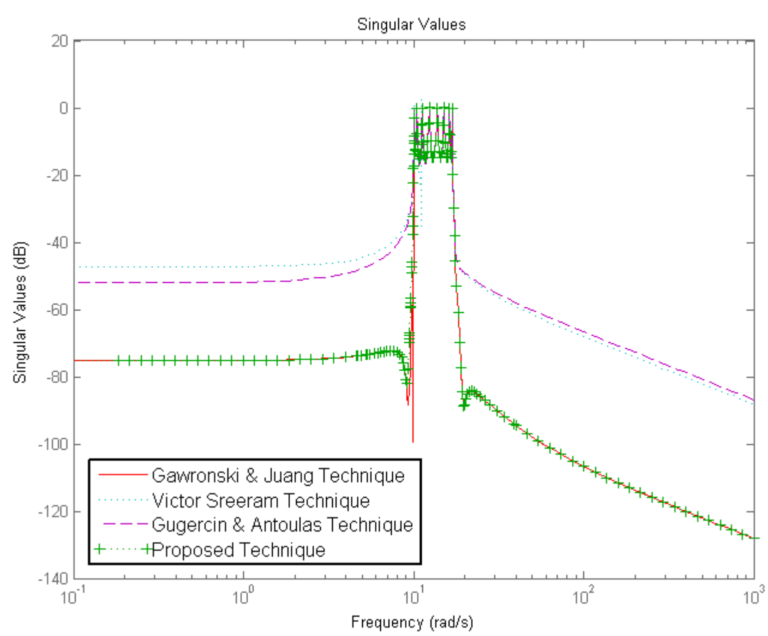

Figure 4.23: Error comparison in entire frequency range for 4^{th} order LOAS

For 5^{th} order LOAS:

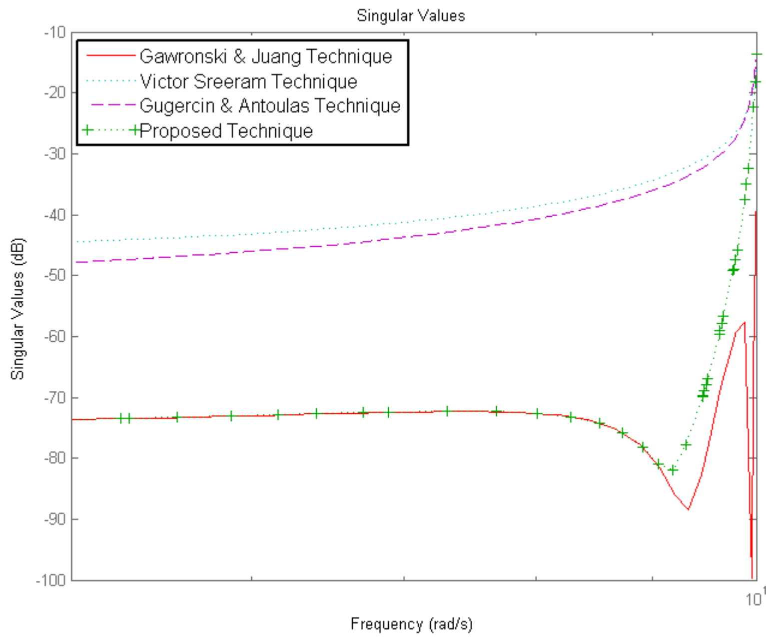

Figure 4.24: Error comparison plot in pin point frequency interval for 4^{th} order LOAS

$$G_{rgj}(s) = \frac{0.00224s^4 - 0.02487s^3 + 0.2685s^2 - 1.924s + 7.271}{s^5 + 2.424s^4 + 203.6s^3 + 494.7s^2 + 1.036e04s + 2.523e^4}$$

$$G_{rvs}(s) = \frac{-0.01568s^4 + 0.4862s^3 + 0.96s^2 + 50.04s + 325.9}{s^5 + 0.07704s^4 + 226.7s^3 + 11.6s^2 + 1.273e04s + 336.3}$$

$$G_{rga}(s) = \frac{-0.02947s^4 + 0.2796s^3 - 1.347s^2 + 27.66s + 236.2}{s^5 + 0.07083s^4 + 230.5s^3 + 9.807s^2 + 1.312e^4s + 200.8}$$

$$G_{rc}(s) = \frac{0.00224s^4 - 0.02487s^3 + 0.2685s^2 - 1.924s + 7.271}{s^5 + 2.59s^4 + 204s^3 + 511.6s^2 + 1.04e^4s + 2.523e^4}$$

For 6^{th} order LOAS:

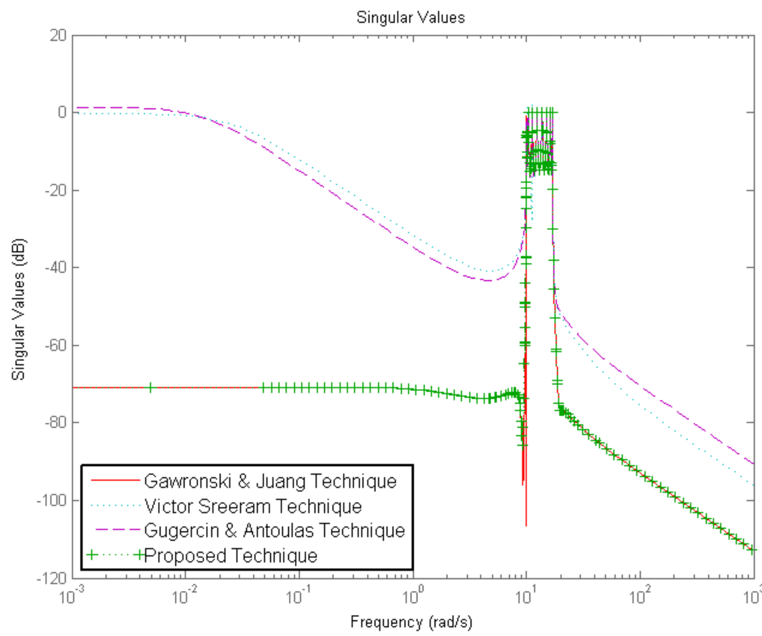

Figure 4.25: Error comparison in entire frequency range for 5^{th} order LOAS

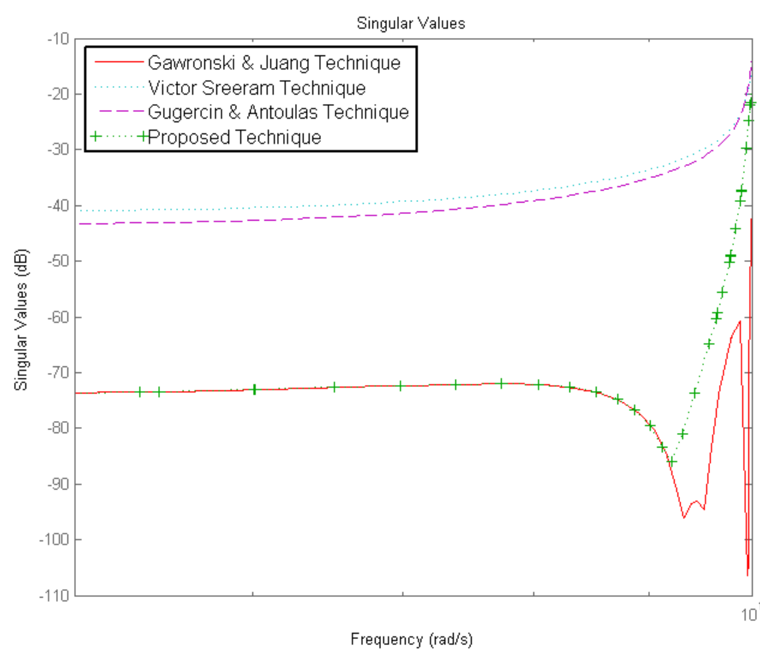

Figure 4.26: Error comparison plot in pin point frequency interval for 5^{th} order LOAS

$$G_{rgj}(s) = \frac{-8.205e-05s^5 + 0.006159s^4 - 0.01017s^3 + 0.7045s^2 - 0.3223s + 21}{s^6 + 0.05274s^5 + 312.9s^4 + 11.42s^3 + 3.267e^4s^2 + 616.5s + 1.138e^6}$$

$$G_{rvs}(s) = \frac{-0.02851s^5 + 0.9971s^4 - 9.821s^3 + 265s^2 - 762.5s + 1.673e^4}{s^6 + 0.08145s^5 + 416.9s^4 + 22.67s^3 + 5.581e^4s^2 + 1506s + 2.417e^6}$$

$$G_{rga}(s) = \frac{-0.05106s^5 + 0.8178s^4 - 15.97s^3 + 214.9s^2 - 1163s + 1.337e^4}{s^6 + 0.09016s^5 + 427.5s^4 + 25.8s^3 + 5.846e^4s^2 + 1755s + 2.577e^6}$$

$$G_{rc}(s) = \frac{-8.205e^{-5}s^5 + 0.006159s^4 - 0.01017s^3 + 0.7045s^2 - 0.3223s + 21}{s^6 + 1.15s^5 + 313.6s^4 + 238.3s^3 + 3.274e^4s^2 + 1.234e^4s + 1.138e^6}$$

For 7^{th} order LOAS:

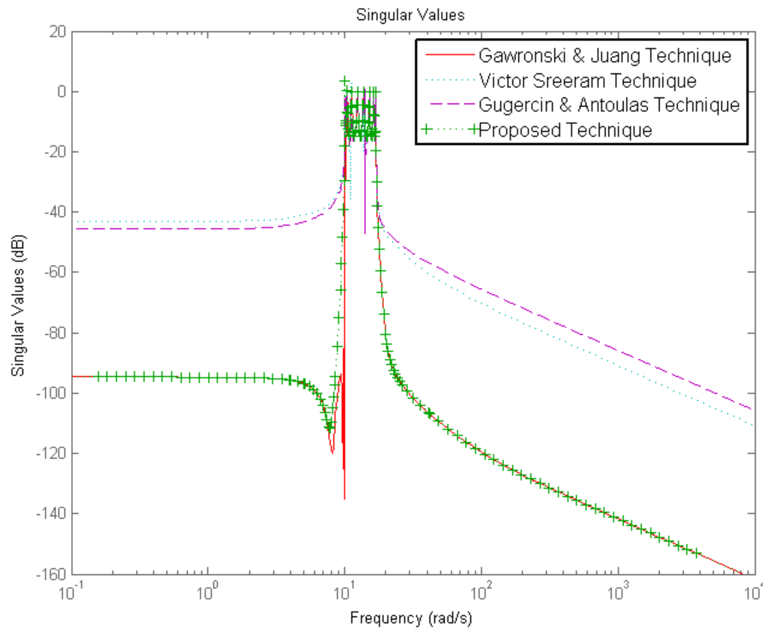

Figure 4.27: Error comparison in entire frequency range for 6^{th} order LOAS

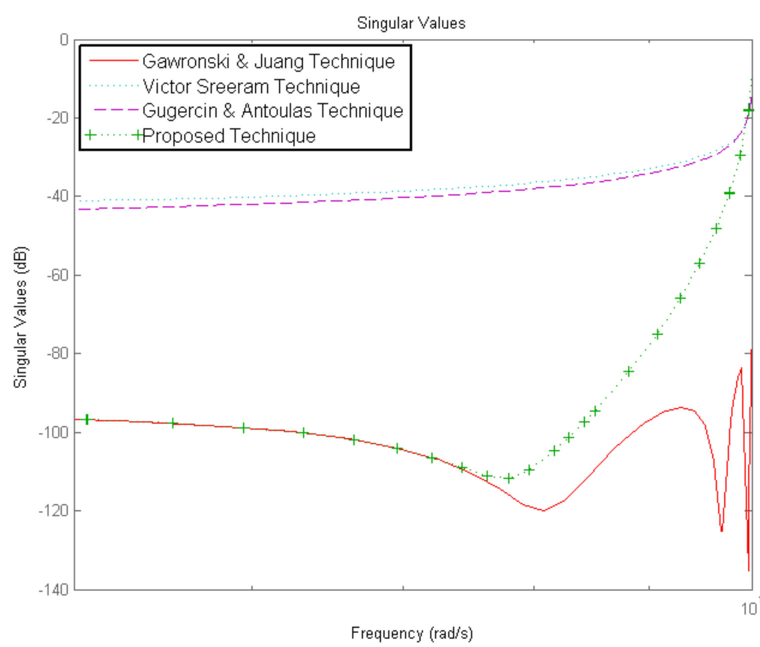

Figure 4.28: Error comparison plot in pin point frequency interval for 6^{th} order LOAS

$$G_{rgj}(s) = \frac{\begin{array}{r} -0.0006978s^6 + 0.006852s^5 - 0.1057s^4 + 0.8031s^3 \\ -5.288s^2 + 24.41s - 82.52 \end{array}}{\begin{array}{r} s^7 + 3.918s^6 + 314.5s^5 + 1246s^4 + 3.3e^4s^3 \\ +1.32e^5s^2 + 1.155e^6s + 4.658e^6 \end{array}}$$

$$G_{rvs}(s) = \frac{\begin{array}{r} -0.02696s^6 + 1.004s^5 - 9.092s^4 + 266.9s^3 \\ -649.9s^2 + 1.684e^4s + 5633 \end{array}}{\begin{array}{r} s^7 + 0.08994s^6 + 418s^5 + 26.02s^4 + 5.607e^4s^3 \\ +1926s^2 + 2.432e^6s + 1.69e^4 \end{array}}$$

$$G_{rga}(s) = \frac{\begin{array}{r} 0.09527s^6 + 0.2456s^5 + 51.67s^4 + 58.58s^3 \\ +8694s^2 + 3177s + 4.578e^5 \end{array}}{\begin{array}{r} s^7 + 0.2162s^6 + 438.3s^5 + 84.42s^4 + 6.12e04s^3 \\ +1.027e^4s^2 + 2.744e^6s + 3.922e^5 \end{array}}$$

$$G_{rc}(s) = \frac{\begin{array}{r} -0.0006978s^6 + 0.006852s^5 - 0.1057s^4 + 0.8031s^3 \\ -5.288s^2 + 24.41s - 82.52 \end{array}}{\begin{array}{r} s^7 + 5.134s^6 + 320s^5 + 1500s^4 + 3.408e^4s^3 \\ +1.453e^5s^2 + 1.207e^6s + 4.658e^6 \end{array}}$$

For 8^{th} order LOAS:

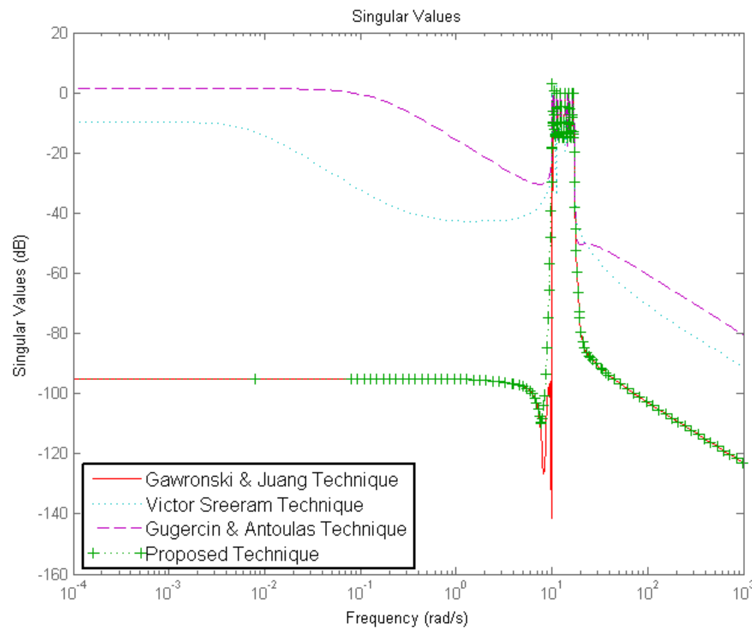

Figure 4.29: Error comparison in entire frequency range for 7^{th} order LOAS

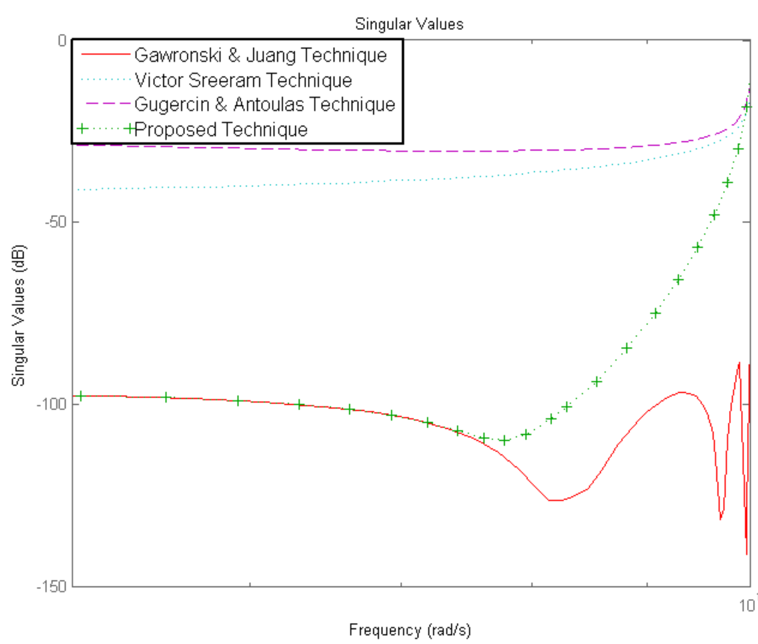

Figure 4.30: Error comparison plot in pin point frequency interval for 7^{th} order LOAS

$$G_{rgj}(s) = \frac{\begin{array}{r} 2.288e^{-5}s^7 - 0.002493s^6 + 0.003145s^5 - 0.3016s^4 \\ +0.1539s^3 - 13.49s^2 + 2.583s - 210.5 \end{array}}{\begin{array}{r} s^8 + 0.1131s^7 + 438.3s^6 + 38.82s^5 + 7.218e^4s^4 \\ +4429s^3 + 5.291e^6s^2 + 1.679e^5s + 1.456e^8 \end{array}}$$

$$G_{rvs}(s) = \frac{\begin{array}{r} 0.1146s^7 + 0.459s^6 + 49.75s^5 + 317.8s^4 \\ +6644s^3 + 6.007e^4s^2 + 2.759e^5s + 3.331e^6 \end{array}}{\begin{array}{r} s^8 + 0.2107s^7 + 638.7s^6 + 99.34s^5 + 1.469e^5s^4 \\ +1.489e^4s^3 + 1.446e^7s^2 + 7.137e^5s + 5.164e^8 \end{array}}$$

$$G_{rga}(s) = \frac{\begin{array}{r} 0.09562s^7 + 0.9509s^6 + 39.42s^5 + 540.4s^4 \\ +4905s^3 + 9.186e^4s^2 + 1.847e^5s + 4.777e^6 \end{array}}{\begin{array}{r} s^8 + 0.2319s^7 + 648.2s^6 + 111.5s^5 + 1.512e^5s^4 \\ +1.703e^4s^3 + 1.508e^7s^2 + 8.294e^5s + 5.448e^8 \end{array}}$$

$$G_{rc}(s) = \frac{\begin{array}{r} 2.288e^{-5}s^7 - 0.002493s^6 + 0.003145s^5 - 0.3016s^4 \\ +0.1539s^3 - 13.49s^2 + 2.583s - 210.5 \end{array}}{\begin{array}{r} s^8 + 2.732s^7 + 442s^6 + 887.6s^5 + 7.296e^4s^4 \\ +9.599e^4s^3 + 5.331e^6s^2 + 3.455e^6s + 1.456e^8 \end{array}}$$

For 9^{th} order LOAS:

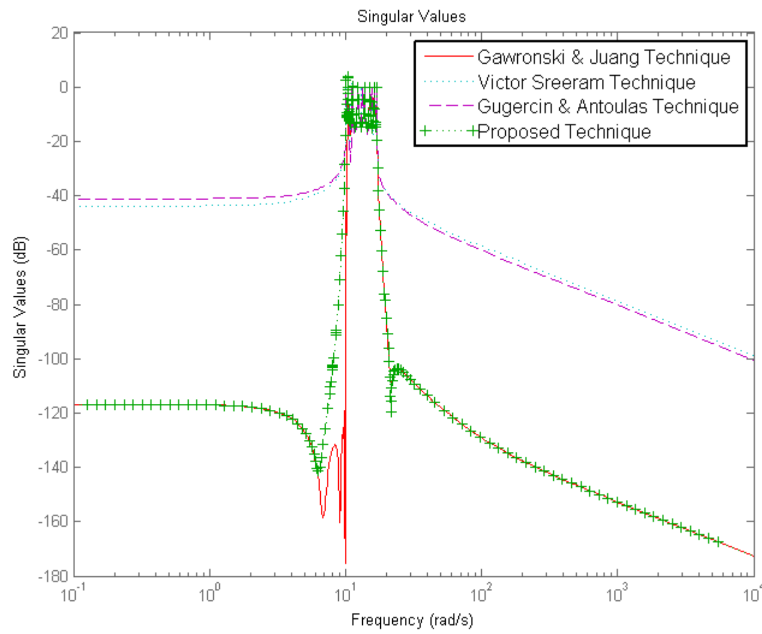

Figure 4.31: Error comparison in entire frequency range for 8^{th} order LOAS

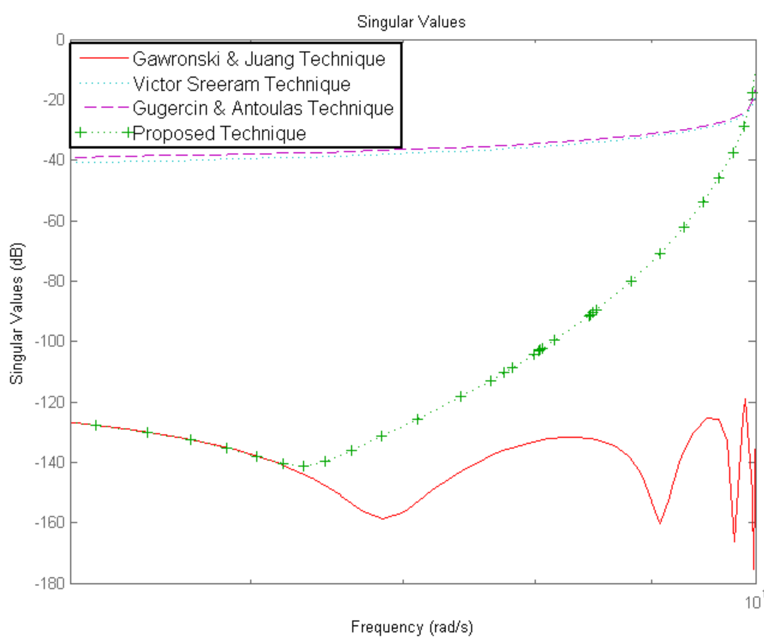

Figure 4.32: Error comparison plot in pin point frequency interval for 8^{th} order LOAS

$$G_{rgj}(s) = \frac{\begin{array}{r} 0.0002851s^8 - 0.002648s^7 + 0.0422s^6 - 0.3299s^5 \\ +2.665s^4 - 15.2s^3 + 75.93s^2 \\ -244.1s + 792.3 \end{array}}{\begin{array}{r} s^9 + 6.134s^8 + 441.3s^7 + 2759s^6 + 7.317e^4s^5 \\ +4.652e^5s^4 + 5.4e^6s^3 + 3.485e^7s^2 \\ +1.496e^8s + 9.781e^8 \end{array}}$$

$$G_{rvs}(s) = \frac{\begin{array}{r} -0.1168s^8 + 2.105s^7 - 97.54s^6 + 1072s^5 \\ -2.647e^4s^4 + 1.69e^5s^3 - 2.858e^6s^2 + 8.342e^6s \\ -1.06e^8 \end{array}}{\begin{array}{r} s^9 + 0.4741s^8 + 655s^7 + 271.9s^6 + 1.543e^5s^5 \\ +5.502e^4s^4 + 1.552e^7s^3 + 4.659e^6s^2 + 5.645e^8s \\ +1.393e^8 \end{array}}$$

$$G_{rga}(s) = \frac{\begin{array}{r} -0.1269s^8 + 2.304s^7 - 102.8s^6 + 1160s^5 \\ -2.733e^4s^4 + 1.815e^5s^3 - 2.9e^6s^2 + 8.908e^6s \\ -1.059e^8 \end{array}}{\begin{array}{r} s^9 + 0.4469s^8 + 660.1s^7 + 249.8s^6 + 1.566e^5s^5 \\ +4.861e^4s^4 + 1.585e^7s^3 + 3.878e^6s^2 \\ +5.8e^8s + 1.057e^8 \end{array}}$$

$$G_{rc}(s) = \frac{\begin{array}{r} 0.0002851s^8 - 0.002648s^7 + 0.0422s^6 - 0.3299s^5 \\ +2.665s^4 - 15.2s^3 + 75.93s^2 - \\ 244.1s + 792.3 \end{array}}{\begin{array}{r} s^9 + 9.073s^8 + 463.7s^7 + 3739s^6 + 8.006e^4 s^5 \\ +5.738e^5 s^4 + 6.107e^6 s^3 + 3.884e^7 s^2 + \\ 1.737e^8 s + 9.781e^8 \end{array}}$$

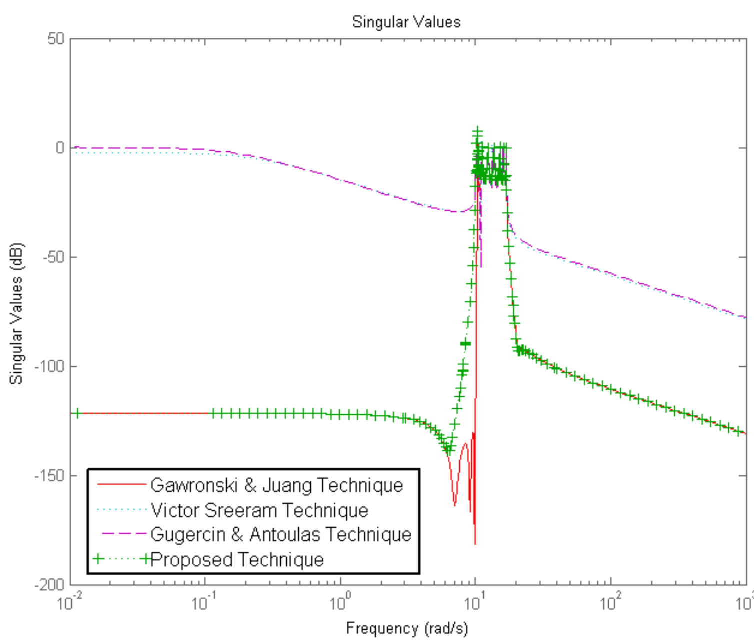

Figure 4.33: Error comparison in entire frequency range for 9^{th} order LOAS

For 10^{th} order LOAS:

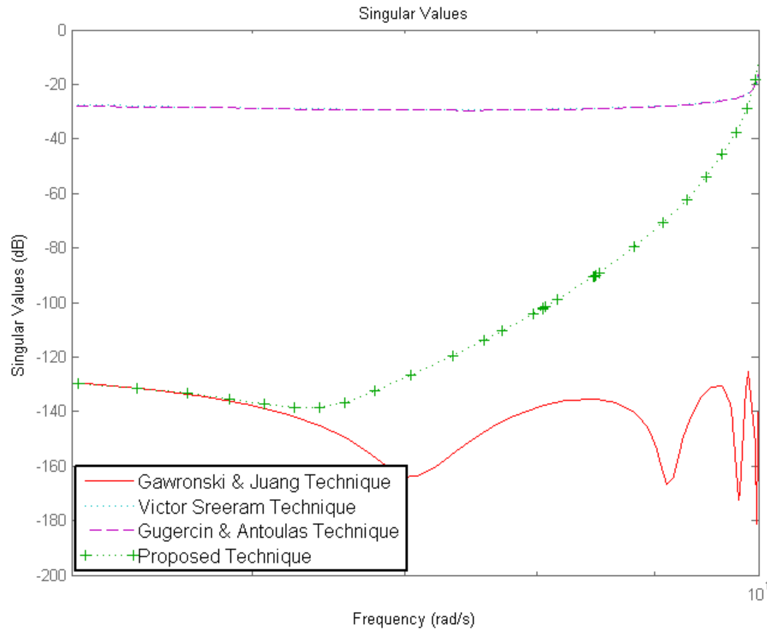

Figure 4.34: Error comparison plot in pin point frequency interval for 9^{th} order LOAS

$$G_{rgj}(s) = \frac{\begin{array}{r} -8.075e^{-6}s^9 + 0.001304s^8 - 0.000966s^7 + 0.1164s^6 \\ -0.05621s^5 + 6.153s^4 - 1.551s^3 + 157.7s^2 \\ -16.55s + 1581 \end{array}}{\begin{array}{r} s^{10} + 0.211s^9 + 593.7s^8 + 104.7s^7 + 1.414e^5s^6 \\ +1.942e^4s^5 + 1.686e^7s^4 + 1.592e^6s^3 \\ +1.006e^9s^2 + 4.868e^7s + 2.399e^{10} \end{array}}$$

$$G_{rvs}(s) = \frac{\begin{array}{r} -0.07711s^9 - 0.6303s^8 - 40.03s^7 - 573.9s^6 - 6419s^5 \\ -1.634e^5s^4 - 3.279e^5s^3 - 1.817e^7s^2 \\ -9.834e05s - 6.905e^8 \end{array}}{\begin{array}{r} s^{10} + 0.4907s^9 + 844.2s^8 + 327.6s^7 + 2.754e^5s^6 \\ +7.869e^4s^5 + 4.34e^7s^4 + 8.076e^6s^3 + 3.312e09s^2 \\ +2.998e^8s + 9.81e^{10} \end{array}}$$

$$G_{rga}(s) = \frac{\begin{array}{r} -0.06721s^9 - 0.9587s^8 - 31.95s^7 - 781.4s^6 - 4158s^5 \\ -2.1e^5s^4 - 6.85e^4s^3 - 2.259e^7s^2 \\ +9.465e^6s - 8.413e^8 \end{array}}{\begin{array}{r} s^{10} + 0.5127s^9 + 848.9s^8 + 344.7s^7 + 2.784e^5s^6 \\ +8.336e^4s^5 + 4.409e^7s^4 + 8.606e^6s^3 \\ +3.378e^9s^2 + 3.21e^8s + 1.004e^{11} \end{array}}$$

$$G_{rc}(s) = \frac{\begin{array}{r} -8.075e^{-6}s^9 + 0.001304s^8 - 0.000966s^7 + 0.1164s^6 \\ -0.05621s^5 + 6.153s^4 - 1.551s^3 + 157.7s^2 \\ -16.55s + 1581 \end{array}}{\begin{array}{r} s^{10} + 4.853s^9 + 605.5s^8 + 2264s^7 + 1.453e^5s^6 \\ +3.945e^5s^5 + 1.729e^7s^4 + 3.043e^7s^3 \\ +1.022e^9s^2 + 8.769e^8s + 2.399e^{10} \end{array}}$$

For 11^{th} order LOAS:

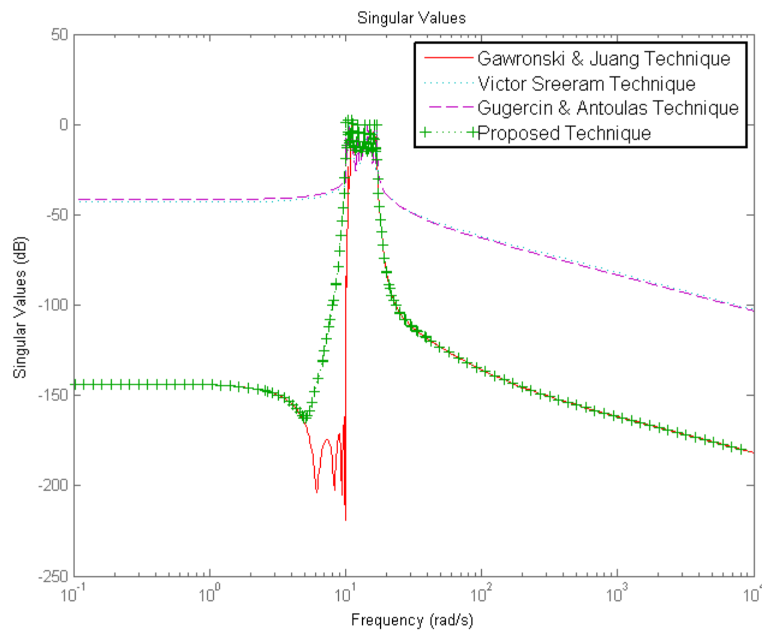

Figure 4.35: Error comparison in entire frequency range for 10^{th} order LOAS

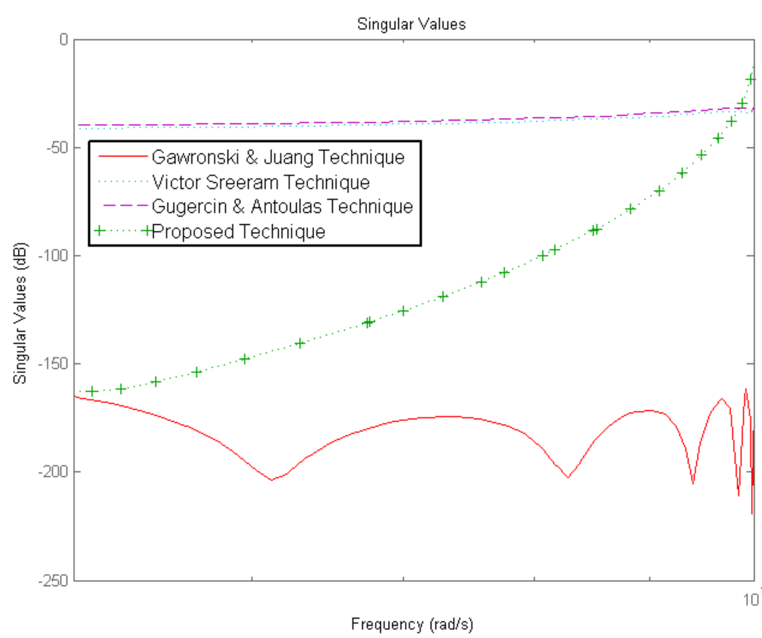

Figure 4.36: Error comparison plot in pin point frequency interval for 10^{th} order LOAS

$$G_{rgj}(s) = \frac{\begin{array}{r} -6.631e^{-5}s^{10} + 0.001368s^9 - 0.006677s^8 + 0.1269s^7 \\ -0.5728s^6 + 6.891s^5 - 22.08s^4 + 181.4s^3 - 437.2s^2 \\ +1871s - 3472 \end{array}}{\begin{array}{r} s^{11} + 4.059s^{10} + 597.1s^9 + 2489s^8 + 1.43e^5s^7 \\ +6.093e^5s^6 + 1.714e^7s^5 + 7.437e^7s^4 + 1.028e^9s^3 \\ +4.524e^9s^2 + 2.464e^{10}s + 1.096e^{11} \end{array}}$$

$$G_{rvs}(s) = \frac{\begin{array}{r} 0.0779s^{10} - 2.142s^9 + 88.87s^8 - 1529s^7 + 3.383e^4s^6 \\ -3.775e^5s^5 + 5.584e^6s^4 - 3.847e^7s^3 \\ +4.1e^8s^2 - 1.383e^9s + 1.088e^{10} \end{array}}{\begin{array}{r} s^{11} + 0.6917s^{10} + 858.4s^9 + 498.8s^8 \\ +2.845e^5s^7 + 1.338e^5s^6 + 4.552e^7s^5 + 1.646e^7s^4 \\ +3.52e^9s^3 + 9.061e^8s^2 + 1.055e^{11}s + 1.675e^{10} \end{array}}$$

$$G_{rga}(s) = \frac{\begin{array}{r} 0.08348s^{10} - 2.25s^9 + 94.26s^8 - 1602s^7 + 3.575e^4s^6 \\ -3.954e^5s^5 + 5.903e^6s^4 - 4.034e^7s^3 \\ +4.349e^8s^2 - 1.453e^9s + 1.162e^{10} \end{array}}{\begin{array}{r} s^{11} + 0.677s^{10} + 859.3s^9 + 483.3s^8 + 2.85e^5s^7 \\ +1.276e^5s^6 + 4.562e^7s^5 + 1.528e^7s^4 + 3.529e^9s^3 \\ +8.009e^8s^2 + 1.058e^{11}s + 1.32e^{10} \end{array}}$$

$$G_{rc}(s) = \frac{\begin{array}{r} -6.631e^{-5}s^{10} + 0.001368s^9 - 0.006677s^8 + 0.1269s^7 \\ -0.5728s^6 + 6.891s^5 - 22.08s^4 + 181.4s^3 \\ -437.2s^2 + 1871s - 3472 \end{array}}{\begin{array}{r} s^{11} + 9.098s^{10} + 630.3s^9 + 4887s^8 + 1.57e^5s^7 \\ +1.035e^6s^6 + 1.935e^7s^5 + 1.077e^8s^4 + 1.182e^9s^3 \\ +5.499e^9s^2 + 2.865e^{10}s + 1.096e^{11} \end{array}}$$

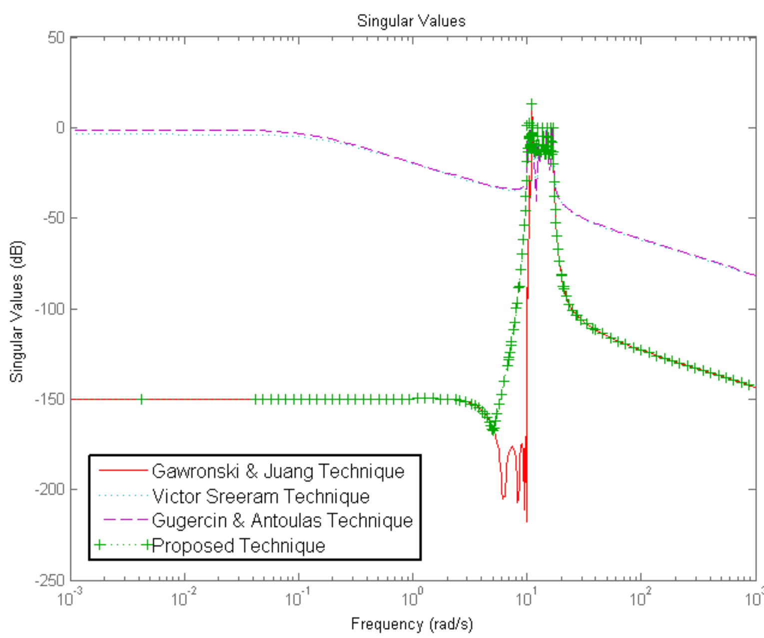

Figure 4.37: Error comparison in entire frequency range for 11^{th} order LOAS

For 12^{th} order LOAS:

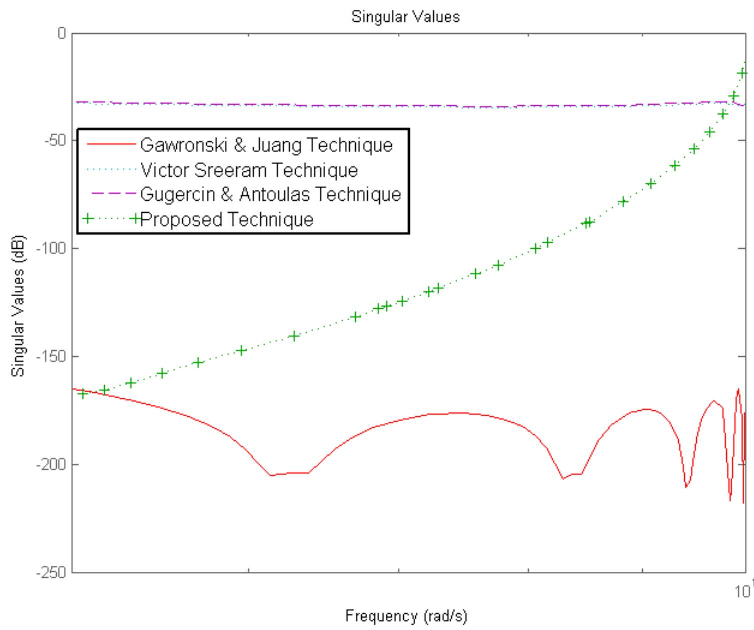

Figure 4.38: Error comparison plot in pin point frequency interval for 11^{th} order LOAS

$$G_{rgj}(s) = \frac{\begin{array}{r} 7.905e^{-6}s^{11} - 0.0007995s^{10} + 0.0007917s^{9} + 0.009321s^{8} \\ +0.07658s^{7} - 4.222s^{6} + 3.242s^{5} - 167.6s^{4} + 71.41s^{3} \\ -3522s^{2} + 641.2s - 3.054e^{4} \end{array}}{\begin{array}{c} s^{12} + 0.1515s^{11} + 814.4s^{10} + 123.3s^{9} + 2.759e^{5}s^{8} \\ +3.823e^{4}s^{7} + 4.966e^{7}s^{6} + 5.711e^{6}s^{5} + 5.004e^{9}s^{4} \\ +4.142e^{8}s^{3} + 2.672e^{11}s^{2} + 1.172e^{10}s + 5.904e^{12} \end{array}}$$

$$G_{rvs}(s) = \frac{\begin{array}{r} 0.05246s^{11} - 0.3728s^{10} + 42.47s^9 - 162.3s^8 + 1.237e^4 s^7 \\ -7664s^6 + 1.623e^6 s^5 + 4.392e^6 s^4 + 9.702e^7 s^3 \\ +6.361e^8 s^2 + 2.115e^9 s + 2.464e^{10} \end{array}}{\begin{array}{c} s^{12} + 0.7086s^{11} + 1035s^{10} + 604.8s^9 + 4.335e^5 s^8 \\ +1.994e^5 s^7 + 9.402e^7 s^6 + 3.178e^7 s^5 + 1.114e^{10} s^4 \\ +2.451e^9 s^3 + 6.854e^{11} s^2 + 7.339e^{10} s + 1.712e^{13} \end{array}}$$

$$G_{rga}(s) = \frac{\begin{array}{r} 0.05336s^{11} - 0.3066s^{10} + 42.72s^9 - 103.6s^8 + 1.228e^4 s^7 \\ +1.207e^4 s^6 + 1.582e^6 s^5 + 7.547e^6 s^4 + 9.185e^7 s^3 \\ +8.771e^8 s^2 + 1.907e^9 s + 3.171e^{10} \end{array}}{\begin{array}{c} s^{12} + 0.7123s^{11} + 1036s^{10} + 608.6s^9 + 4.343e^5 s^8 \\ +2.009e^5 s^7 + 9.425e^7 s^6 + 3.203e^7 s^5 + 1.118e^{10} s^4 \\ +2.472e^9 s^3 + 6.879e^{11} s^2 + 7.404e^{10} s + 1.719e^{13} \end{array}}$$

$$G_{rc}(s) = \frac{\begin{array}{r} 7.905e^{-6} s^{11} - 0.0007995s^{10} + 0.0007917s^9 + 0.009321s^8 \\ +0.07658s^7 - 4.222s^6 + 3.242s^5 - 167.6s^4 + 71.41s^3 \\ -3522s^2 + 641.2s - 3.054e^4 \end{array}}{\begin{array}{c} s^{12} + 6.2s^{11} + 833.6s^{10} + 4062s^9 + 2.852e^5 s^8 \\ +1.055e^6 s^7 + 5.135e^7 s^6 + 1.356e^8 s^5 + 5.139e^9 s^4 \\ +8.644e^9 s^3 + 2.712e^{11} s^2 + 2.184e^{11} s + 5.904e^{12} \end{array}}$$

For 13^{th} order LOAS:

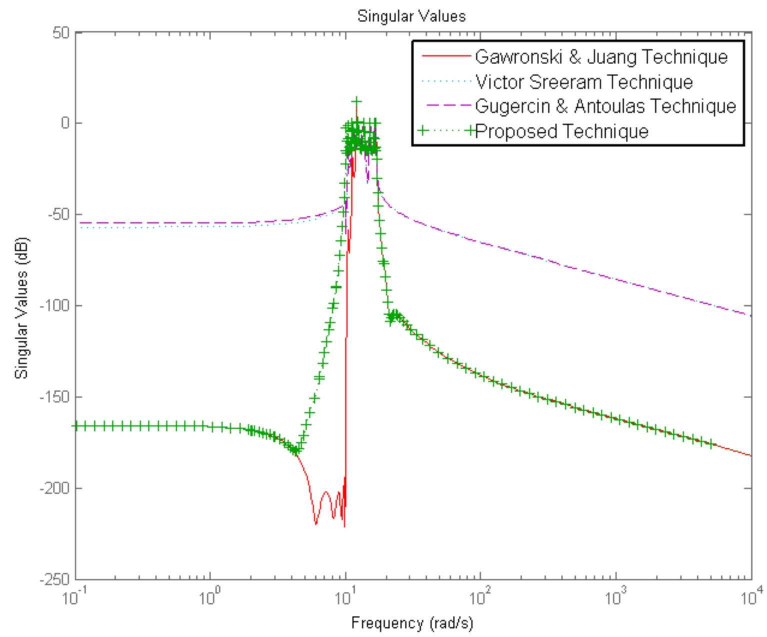

Figure 4.39: Error comparison in entire frequency range for 12^{th} order LOAS

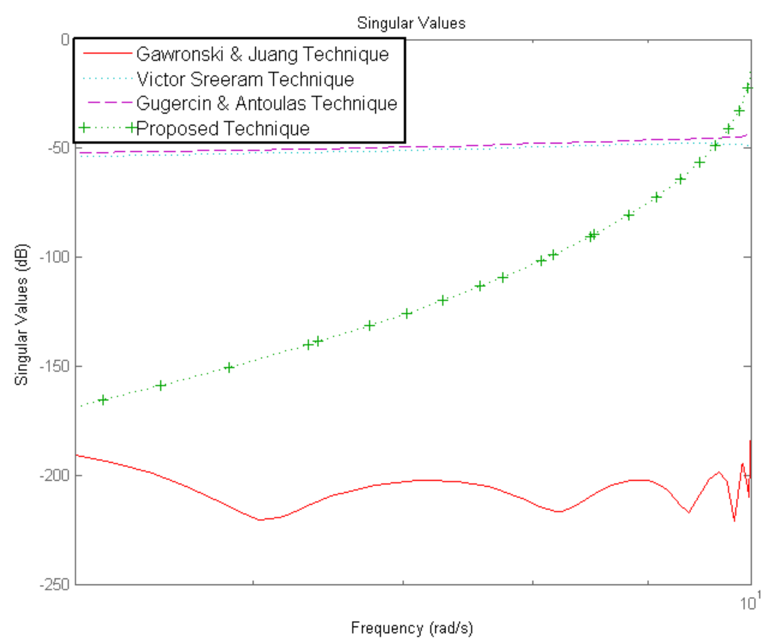

Figure 4.40: Error comparison plot in pin point frequency interval for 12^{th} order LOAS

$$G_{rgj}(s) = \frac{\begin{array}{r} 5.783e^{-5}s^{12} - 0.001467s^{11} + 0.006104s^{10} \\ -0.05755s^{9} + 1.024s^{8} - 10.39s^{7} + 37.53s^{6} - 421.2s^{5} \\ +1067s^{4} - 8947s^{3} + 1.567e^{4}s^{2} - 7.786e^{4}s \\ +8.889e^{4} \end{array}}{\begin{array}{r} s^{13} + 2.8s^{12} + 860.6s^{11} + 2544s^{10} + 3.047e^{5}s^{9} \\ +9.374e^{5}s^{8} + 5.686e^{7}s^{7} + 1.799e^{8}s^{6} + 5.898e^{9}s^{5} \\ +1.903e^{10}s^{4} + 3.227e^{11}s^{3} + 1.055e^{12}s^{2} + 7.274e^{12}s \\ +2.395e^{13} \end{array}}$$

$$G_{rvs}(s) = \frac{\begin{array}{r} -0.01096s^{12} + 0.4963s^{11} - 25.32s^{10} + 555.7s^{9} \\ -1.583e^{4}s^{8} + 2.172e^{5}s^{7} - 4.236e^{6}s^{6} + 3.787e^{7}s^{5} \\ -5.48e^{8}s^{4} + 3.021e^{9}s^{3} - 3.38e^{10}s^{2} + 9.012e^{10}s \\ -7.954e^{11} \end{array}}{\begin{array}{r} s^{13} + 0.7898s^{12} + 1047s^{11} + 695.2s^{10} + 4.435e^{5}s^{9} \\ +2.384e^{5}s^{8} + 9.724e^{7}s^{7} + 4.016e^{7}s^{6} + 1.164e^{10}s^{5} \\ +3.399e^{9}s^{4} + 7.231e^{11}s^{3} + 1.272e^{11}s^{2} + 1.822e^{13}s \\ +1.2e^{12} \end{array}}$$

$$G_{rga}(s) = \frac{\begin{array}{r} -0.01196s^{12} + 0.5465s^{11} - 27.61s^{10} + 607.3s^{9} \\ -1.73e^{4}s^{8} + 2.371e^{5}s^{7} - 4.653e^{6}s^{6} + 4.146e^{7}s^{5} \\ -6.072e^{8}s^{4} + 3.327e^{9}s^{3} - 3.789e^{10}s^{2} + 9.997e^{10}s \\ -9.043e^{11} \end{array}}{\begin{array}{r} s^{13} + 0.7879s^{12} + 1047s^{11} + 691.8s^{10} + 4.43e^{5}s^{9} \\ +2.364e^{5}s^{8} + 9.704e^{7}s^{7} + 3.96e^{7}s^{6} + 1.161e^{10}s^{5} \\ +3.324e^{9}s^{4} + 7.201e^{11}s^{3} + 1.222e^{11}s^{2} + 1.812e^{13}s \\ +1.072e^{12} \end{array}}$$

$$G_{rc}(s) = \frac{\begin{array}{r} 5.783e^{-5}s^{12} - 0.001467s^{11} + 0.006104s^{10} - 0.05755s^{9} \\ +1.024s^{8} - 10.39s^{7} + 37.53s^{6} - 421.2s^{5} + 1067s^{4} \\ -8947s^{3} + 1.567e^{4}s^{2} - 7.786e^{4}s \\ +8.889e^{4} \end{array}}{\begin{array}{r} s^{13} + 6.577s^{12} + 878.3s^{11} + 5148s^{10} + 3.161e^{5}s^{9} \\ +1.644e^{6}s^{8} + 5.97e^{7}s^{7} + 2.743e^{8}s^{6} + 6.25e^{9}s^{5} \\ +2.525e^{10}s^{4} + 3.44e^{11}s^{3} + 1.216e^{12}s^{2} + 7.788e^{12}s \\ +2.395e^{13} \end{array}}$$

For 14^{th} order LOAS:

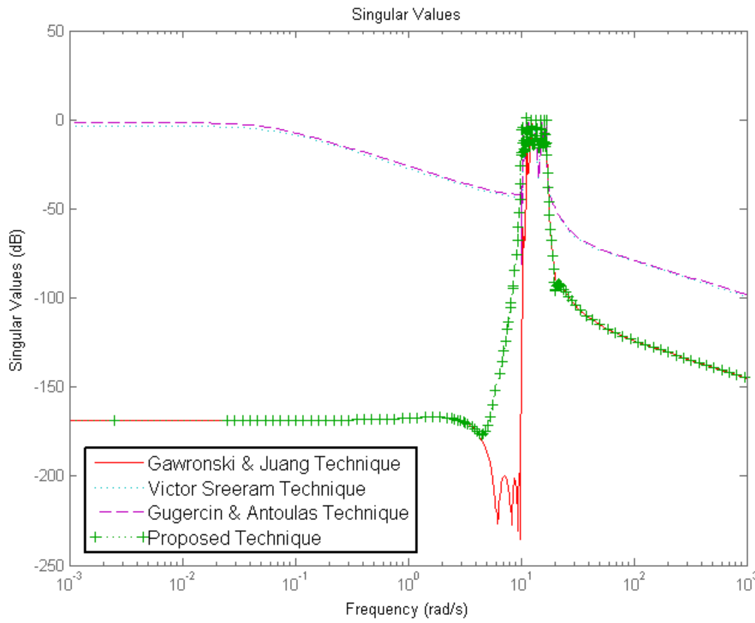

Figure 4.41: Error comparison in entire frequency range for 13^{th} order LOAS

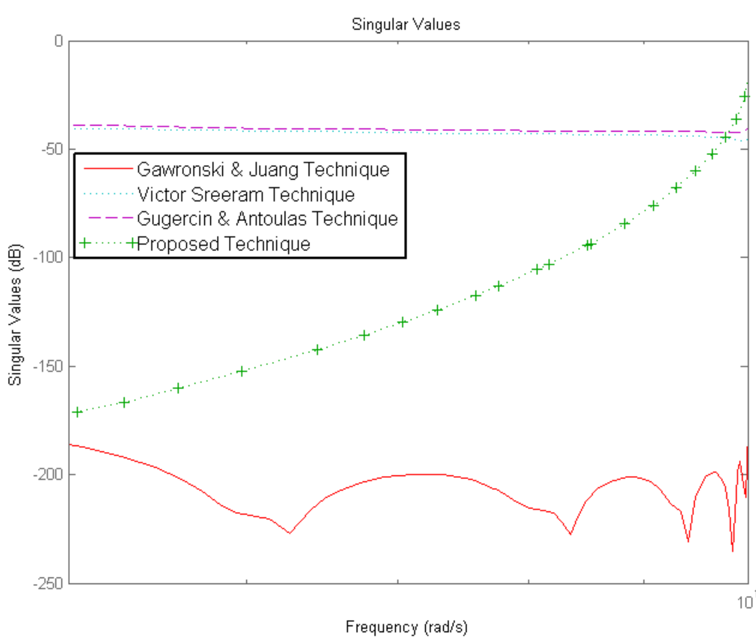

Figure 4.42: Error comparison plot in pin point frequency interval for 13^{th} order LOAS

$$G_{rgj}(s) = \frac{\begin{aligned} -6.984e^{-6}s^{13} + 0.0007537s^{12} - 0.0007541s^{11} - 0.02277s^{10} \\ -0.1416s^{9} + 34.6s^{8} - 5.668s^{7} + 324.2s^{6} - 185.3s^{5} \\ +9840s^{4} - 3295s^{3} + 1.637e^{5}s^{2} - 2.458e^{4}s + 1.149e^{6} \end{aligned}}{\begin{aligned} s^{14} + 0.4486s^{13} + 1175s^{12} + 465.4s^{11} \\ +5.786e^{5}s^{10} + 1.958e^{5}s^{9} + 1.55e^{8}s^{8} + 4.276e^{7}s^{7} \\ +2.441e^{10}s^{6} + 5.113e^{9}s^{5} + 2.26e^{12}s^{4} + 3.18e^{11}s^{3} \\ +1.141e^{14}s^{2} + 8.049e^{12}s + 2.423e^{15} \end{aligned}}$$

$$G_{rvs}(s) = \frac{\begin{aligned} -0.01015s^{13} + 0.2855s^{12} - 13.19s^{11} + 267.3s^{10} \\ -6291s^{9} + 9.581e^{4}s^{8} - 1.44e^{6}s^{7} + 1.697e^{7}s^{6} \\ -1.716e^{8}s^{5} + 1.587e^{9}s^{4} - 1.028e^{10}s^{3} + 7.465e^{10}s^{2} \\ -2.452e^{11}s + 1.378e^{12} \end{aligned}}{\begin{aligned} s^{14} + 0.7899s^{13} + 1230s^{12} + 825.6s^{11} \\ +6.333e^{5}s^{10} + 3.496e^{5}s^{9} + 1.767e^{8}s^{8} + 7.679e^{7}s^{7} \\ +2.887e^{10}s^{6} + 9.234e^{9}s^{5} + 2.764e^{12}s^{4} + 5.771e^{11}s^{3} \\ +1.437e^{14}s^{2} + 1.466e^{13}s + 3.135e^{15} \end{aligned}}$$

$$G_{rga}(s) = \frac{\begin{array}{r} -0.01089s^{13} + 0.3018s^{12} - 14.11s^{11} + 283.4s^{10} - 6730s^{9} \\ +1.021e^{5}s^{8} - 1.546e^{6}s^{7} + 1.821e^{7}s^{6} - 1.852e^{8}s^{5} \\ +1.718e^{9}s^{4} - 1.117e^{10}s^{3} + 8.166e^{10}s^{2} \\ -2.684e^{11}s + 1.526e^{12} \end{array}}{\begin{array}{r} s^{14} + 0.7902s^{13} + 1230s^{12} + 825.7s^{11} \\ +6.331e^{5}s^{10} + 3.496e^{5}s^{9} + 1.766e^{8}s^{8} + 7.677e^{7}s^{7} \\ +2.885e^{10}s^{6} + 9.23e^{9}s^{5} + 2.762e^{12}s^{4} + 5.767e^{11}s^{3} \\ +1.436e^{14}s^{2} + 1.465e^{13}s + 3.131e^{15} \end{array}}$$

$$G_{rc}(s) = \frac{\begin{array}{r} -6.984e^{-6}s^{13} + 0.0007537s^{12} - 0.0007541s^{11} - 0.02277s^{10} \\ -0.1416s^{9} + 34.6s^{8} - 5.668s^{7} + 324.2s^{6} \\ -185.3s^{5} + 9840s^{4} - 3295s^{3} + 1.637e^{5}s^{2} \\ -2.458e^{4}s + 1.149e^{6} \end{array}}{\begin{array}{r} s^{14} + 0.5456s^{13} + 1175s^{12} + 555.2s^{11} \\ +5.787e^{5}s^{10} + 2.295e^{5}s^{9} + 1.55e^{8}s^{8} + 4.937e^{7}s^{7} \\ +2.441e^{10}s^{6} + 5.828e^{9}s^{5} + 2.26e^{12}s^{4} + 3.585e^{11}s^{3} \\ +1.141e^{14}s^{2} + 8.989e1^{2}s + 2.423e^{15} \end{array}}$$

For 15^{th} order LOAS:

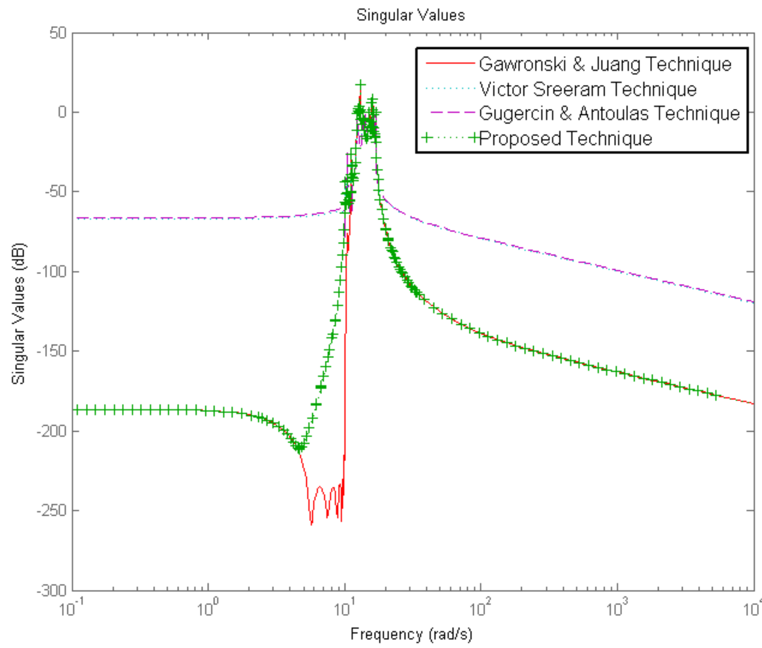

Figure 4.43: Error comparison in entire frequency range for 14^{th} order LOAS

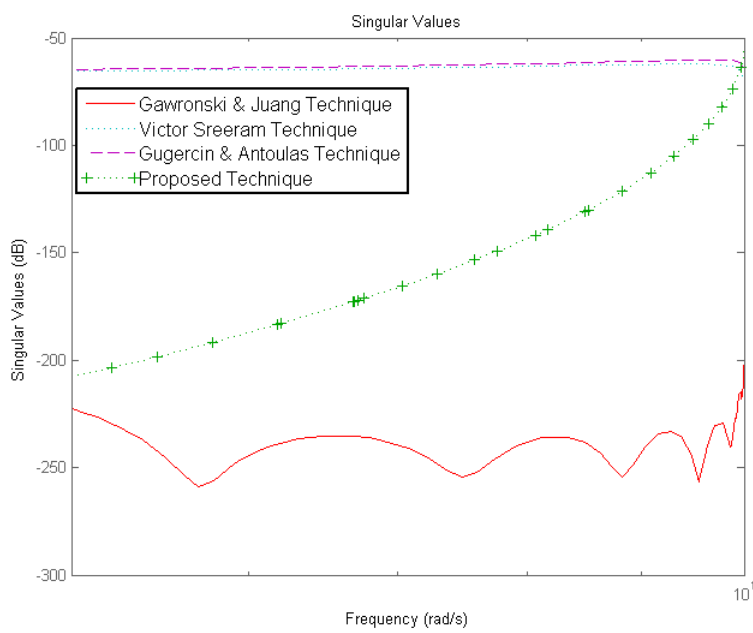

Figure 4.44: Error comparison plot in pin point frequency interval for 14^{th} order LOAS

$$G_{rgj}(s) = \frac{\begin{array}{r} -2.865e^{-5}s^{14} + 0.0008359s^{13} - 0.004622s^{12} - 0.01176s^{11} \\ -0.9393s^{10} + 36.46s^{9} + 0.8053s^{8} + 409.6s^{7} \\ -2214s^{6} + 1.277e^{4}s^{5} - 5.959e^{4}s^{4} + 2.182e^{5}s^{3} \\ -8.915e^{5}s^{2} + 1.572e^{6}s - 5.676e^{6} \end{array}}{\begin{array}{r} s^{15} + 2.021s^{14} + 1180s^{13} + 2359s^{12} \\ +5.834e^{5}s^{11} + 1.151e^{6}s^{10} + 1.569e^{8}s^{9} + 3.041e^{8}s^{8} \\ +2.478e^{10}s^{7} + 4.704e^{10}s^{6} + 2.301e^{12}s^{5} + 4.266e^{12}s^{4} \\ +1.164e^{14}s^{3} + 2.102e^{14}s^{2} + 2.48e^{15}s + 4.349e^{15} \end{array}}$$

$$G_{rvs}(s) = \frac{\begin{array}{r} -0.01144s^{14} + 0.3022s^{13} - 14.81s^{12} + 284.4s^{11} \\ -7124s^{10} + 1.028e^{5}s^{9} - 1.669e^{6}s^{8} + 1.846e^{7}s^{7} \\ -2.076e^{8}s^{6} + 1.76e^{9}s^{5} - 1.355e^{10}s^{4} + 8.498e^{10}s^{3} \\ -4.042e^{11}s^{2} + 1.628e^{12}s - 3.193e^{12} \end{array}}{\begin{array}{r} s^{15} + 0.7924s^{14} + 1231s^{13} + 828.7s^{12} \\ +6.335e^{5}s^{11} + 3.512e^{5}s^{10} + 1.768e^{8}s^{9} + 7.725e^{7}s^{8} \\ +2.889e^{10}s^{7} + 9.308e^{9}s^{6} + 2.766e^{12}s^{5} + 5.84e^{11}s^{4} \\ +1.439e^{14}s^{3} + 1.501e^{13}s^{2} + 3.138e^{15}s + 7.231e^{12} \end{array}}$$

$$G_{rga}(s) = \frac{\begin{array}{r} -0.01209s^{14} + 0.3142s^{13} - 15.61s^{12} + 296.2s^{11} \\ -7511s^{10} + 1.075e^{5}s^{9} - 1.763e^{6}s^{8} + 1.937e^{7}s^{7} \\ -2.201e^{8}s^{6} + 1.854e^{9}s^{5} - 1.443e^{10}s^{4} + 8.997e^{10}s^{3} \\ -4.315e^{11}s^{2} + 1.731e^{12}s - 3.392e^{12} \end{array}}{\begin{array}{r} s^{15} + 0.7919s^{14} + 1230s^{13} + 827.8s^{12} \\ +6.332e^{5}s^{11} + 3.507e^{5}s^{10} + 1.767e^{8}s^{9} + 7.708e^{7}s^{8} \\ +2.886e^{10}s^{7} + 9.279e^{9}s^{6} + 2.763e^{12}s^{5} + 5.813e^{11}s^{4} \\ +1.437e^{14}s^{3} + 1.489e^{13}s^{2} + 3.133e^{15}s + 4.931e^{12} \end{array}}$$

$$G_{rc}(s) = \frac{\begin{array}{r} -2.865e^{-5}s^{14} + 0.0008359s^{13} - 0.004622s^{12} - 0.01176s^{11} \\ -0.9393s^{10} + 36.46s^{9} + 0.8053s^{8} + 409.6s^{7} \\ -2214s^{6} + 1.277e^{4}s^{5} - 5.959e^{4}s^{4} + 2.182e^{5}s^{3} \\ -8.915e^{5}s^{2} + 1.572e^{6}s - 5.676e^{6} \end{array}}{\begin{array}{r} s^{15} + 2.624s^{14} + 1181s^{13} + 2954s^{12} \\ +5.848e^{5}s^{11} + 1.388e^{6}s^{10} + 1.574e^{8}s^{9} + 3.533e^{8}s^{8} \\ +2.488e^{10}s^{7} + 5.263e^{10}s^{6} + 2.312e^{12}s^{5} + 4.596e^{12}s^{4} \\ +1.171e^{14}s^{3} + 2.182e^{14}s^{2} + 2.494e^{15}s + 4.349e^{15} \end{array}}$$

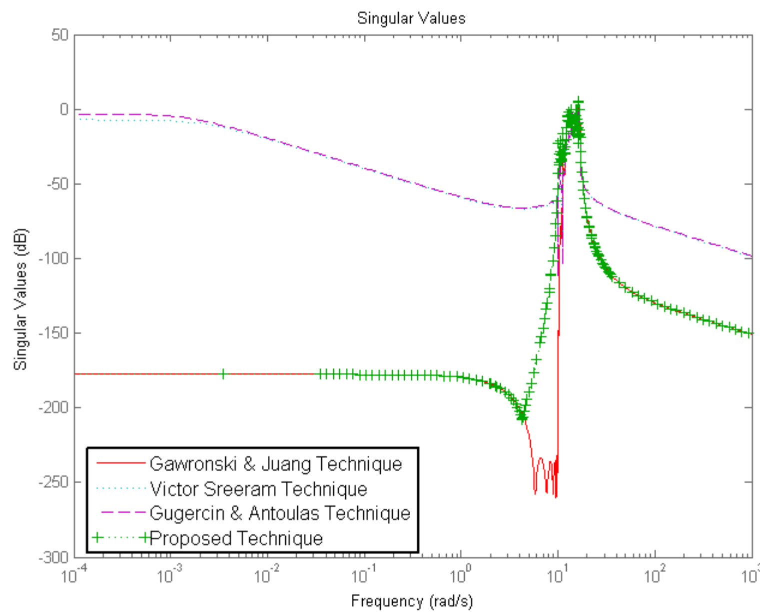

Figure 4.45: Error comparison in entire frequency range for 15^{th} order LOAS

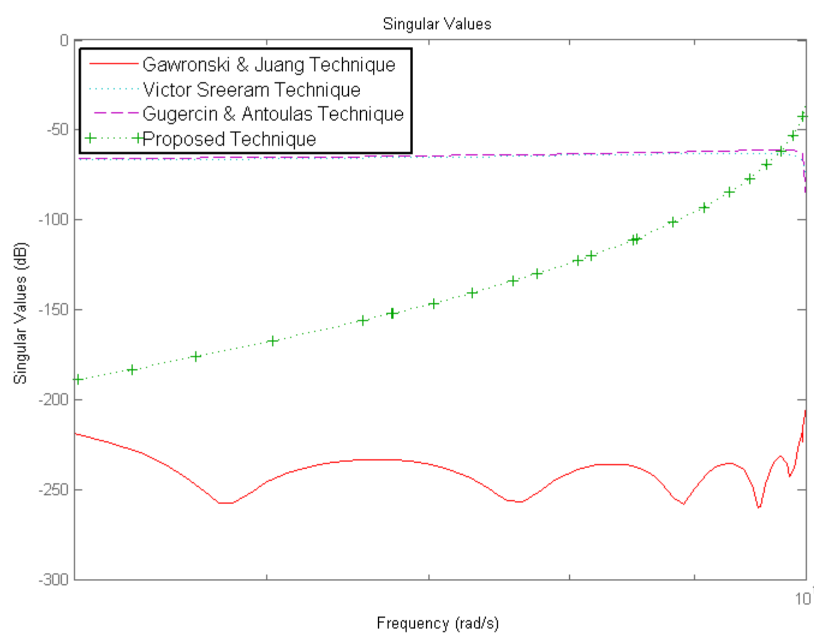

Figure 4.46: Error comparison plot in pin point frequency interval for 15^{th} order LOAS

Table 4.1: 1^{st} Order LOAS

Techniques	LOAS
GJ	$\frac{0.003444}{s+0.1387}$
GA	$\frac{-0.01705}{s+0.01371}$
VS	$\frac{-0.002426}{s+0.005066}$
PT	$\frac{0.003444}{s+0.1387}$

Table 4.2: 2^{nd} Order LOAS

Techniques	LOAS
GJ	$\frac{-0.0017s+0.08309}{s^2+0.009129s+100.1}$
GA	$\frac{-0.02136s-0.04411}{s^2+0.0203s+102.4}$
VS	$\frac{-0.02529s+0.009708}{s^2+0.01801s+102.1}$
PT	$\frac{-0.0017s+0.08309}{s^2+0.009129s+100.1}$

Table 4.3: 3^{rd} Order LOAS

Techniques	LOAS
GJ	$\frac{-0.006734s^2+0.08279s-0.3869}{s^3+1.076s^2+100.2s+107}$
GA	$\frac{-0.01314s^2-0.04678s+0.8388}{s^3+0.02879s^2+102.4s+0.8765}$
VS	$\frac{-0.009374s^2+0.002604s+1.617}{s^3+0.04225s^2+102.1s+2.514}$
PT	$\frac{-0.006734s^2+0.08279s-0.3869}{s^3+1.076s^2+100.2s+107}$

Table 4.4: 4^{th} Order LOAS

Techniques	LOAS
GJ	$\frac{0.0003919s^3-0.02315s^2+0.03128s-1.785}{s^4+0.02109s^3+203s^2+2.188s+1.03e^4}$
GA	$\frac{-0.04579s^3+0.3244s^2-5.316s+32.48}{s^4+0.05703s^3+229.7s^2+6.421s+1.304e^4}$
VS	$\frac{-0.03895s^3+0.5305s^2-4.601s+54.79}{s^4+0.05388s^3+226.2s^2+5.948s+1.267e^4}$
PT	$\frac{0.0003919s^3-0.02315s^2+0.03128s-1.785}{s^4+0.2s^3+203s^2+20.3s+1.03e^4}$

Table 4.5: 5^{th} Order LOAS

Techniques	LOAS
GJ	$\frac{0.00224s^4-0.02487s^3+0.2685s^2-1.924s+7.271}{s^5+2.424s^4+203.6s^3+494.7s^2+1.036e04s+2.523e^4}$
GA	$\frac{-0.02947s^4+0.2796s^3-1.347s^2+27.66s+236.2}{s^5+0.07083s^4+230.5s^3+9.807s^2+1.312e^4s+200.8}$
VS	$\frac{-0.01568s^4+0.4862s^3+0.96s^2+50.04s+325.9}{s^5+0.07704s^4+226.7s^3+11.6s^2+1.273e04s+336.3}$
PT	$\frac{0.00224s^4-0.02487s^3+0.2685s^2-1.924s+7.271}{s^5+2.59s^4+204s^3+511.6s^2+1.04e^4s+2.523e^4}$

Table 4.6: 6^{th} Order LOAS

Techniques	LOAS
GJ	$\frac{-8.205e-05s^5+0.006159s^4-0.01017s^3+0.7045s^2-0.3223s+21}{s^6+0.05274s^5+312.9s^4+11.42s^3+3.267e^4s^2+616.5s+1.138e^6}$
GA	$\frac{-0.05106s^5+0.8178s^4-15.97s^3+214.9s^2-1163s+1.337e^4}{s^6+0.09016s^5+427.5s^4+25.8s^3+5.846e^4s^2+1755s+2.577e^6}$
VS	$\frac{-0.02851s^5+0.9971s^4-9.821s^3+265s^2-762.5s+1.673e^4}{s^6+0.08145s^5+416.9s^4+22.67s^3+5.581e^4s^2+1506s+2.417e^6}$
PT	$\frac{-8.205e^{-5}s^5+0.006159s^4-0.01017s^3+0.7045s^2-0.3223s+21}{s^6+1.15s^5+313.6s^4+238.3s^3+3.274e^4s^2+1.234e^4s+1.138e^6}$

Table 4.7: 7^{th} Order LOAS

Techniques	LOAS
GJ	$\frac{-0.0006978s^6+0.006852s^5-0.1057s^4+0.8031s^3-5.288s^2+24.41s-82.52}{s^7+3.918s^6+314.5s^5+1246s^4+3.3e^4s^3+1.32e^5s^2+1.155e^6s+4.658e^6}$
GA	$\frac{0.09527s^6+0.2456s^5+51.67s^4+58.58s^3+8694s^2+3177s+4.578e^5}{s^7+0.2162s^6+438.3s^5+84.42s^4+6.12e04s^3+1.027e^4s^2+2.744e^6s+3.922e^5}$
VS	$\frac{-0.02696s^6+1.004s^5-9.092s^4+266.9s^3-649.9s^2+1.684e^4s+5633}{s^7+0.08994s^6+418s^5+26.02s^4+5.607e^4s^3+1926s^2+2.432e^6s+1.69e^4}$
PT	$\frac{-0.0006978s^6+0.006852s^5-0.1057s^4+0.8031s^3-5.288s^2+24.41s-82.52}{s^7+5.134s^6+320s^5+1500s^4+3.408e^4s^3+1.453e^5s^2+1.207e^6s+4.658e^6}$

Table 4.8: 8^{th} Order LOAS

Techniques	LOAS
GJ	$\frac{2.288e^{-5}s^7 - 0.002493s^6 + 0.003145s^5 - 0.3016s^4 + 0.1539s^3 - 13.49s^2 + 2.583s - 210.5}{s^8 + 0.1131s^7 + 438.3s^6 + 38.82s^5 + 7.218e^4s^4 + 4429s^3 + 5.291e^6s^2 + 1.679e^5s + 1.456e^8}$
GA	$\frac{0.09562s^7 + 0.9509s^6 + 39.42s^5 + 540.4s^4 + 4905s^3 + 9.186e^4s^2 + 1.847e^5s + 4.777e^6}{s^8 + 0.2319s^7 + 648.2s^6 + 111.5s^5 + 1.512e^5s^4 + 1.703e^4s^3 + 1.508e^7s^2 + 8.294e^5s + 5.448e^8}$
VS	$\frac{0.1146s^7 + 0.459s^6 + 49.75s^5 + 317.8s^4 + 6644s^3 + 6.007e^4s^2 + 2.759e^5s + 3.331e^6}{s^8 + 0.2107s^7 + 638.7s^6 + 99.34s^5 + 1.469e^5s^4 + 1.489e^4s^3 + 1.446e^7s^2 + 7.137e^5s + 5.164e^8}$
PT	$\frac{2.288e^{-5}s^7 - 0.002493s^6 + 0.003145s^5 - 0.3016s^4 + 0.1539s^3 - 13.49s^2 + 2.583s - 210.5}{s^8 + 2.732s^7 + 442s^6 + 887.6s^5 + 7.296e^4s^4 + 9.599e^4s^3 + 5.331e^6s^2 + 3.455e^6s + 1.456e^8}$

Table 4.9: 9^{th} Order LOAS

Techniques	LOAS
GJ	$\frac{0.0002851s^8 - 0.002648s^7 + 0.0422s^6 - 0.3299s^5 + 2.665s^4 - 15.2s^3 + 75.93s^2 - 244.1s + 792.3}{s^9 + 6.134s^8 + 441.3s^7 + 2759s^6 + 7.317e^4s^5 + 4.652e^5s^4 + 5.4e^6s^3 + 3.485e^7s^2 + 1.496e^8s + 9.781e^8}$
GA	$\frac{-0.1269s^8 + 2.304s^7 - 102.8s^6 + 1160s^5 - 2.733e^4s^4 + 1.815e^5s^3 - 2.9e^6s^2 + 8.908e^6s - 1.059e^8}{s^9 + 0.4469s^8 + 660.1s^7 + 249.8s^6 + 1.566e^5s^5 + 4.861e^4s^4 + 1.585e^7s^3 + 3.878e^6s^2 + 5.8e^8s + 1.057e^8}$
VS	$\frac{-0.1168s^8 + 2.105s^7 - 97.54s^6 + 1072s^5 - 2.647e^4s^4 + 1.69e^5s^3 - 2.858e^6s^2 + 8.342e^6s - 1.06e^8}{s^9 + 0.4741s^8 + 655s^7 + 271.9s^6 + 1.543e^5s^5 + 5.502e^4s^4 + 1.552e^7s^3 + 4.659e^6s^2 + 5.645e^8s + 1.393e^8}$
PT	$\frac{0.0002851s^8 - 0.002648s^7 + 0.0422s^6 - 0.3299s^5 + 2.665s^4 - 15.2s^3 + 75.93s^2 - 244.1s + 792.3}{s^9 + 9.073s^8 + 463.7s^7 + 3739s^6 + 8.006e^4s^5 + 5.738e^5s^4 + 6.107e^6s^3 + 3.884e^7s^2 + 1.737e^8s + 9.781e^8}$

Table 4.10: 9^{th} Order LOAS

Techniques	LOAS
GJ	$\frac{0.0002851s^8 - 0.002648s^7 + 0.0422s^6 - 0.3299s^5 + 2.665s^4 - 15.2s^3 + 75.93s^2 - 244.1s + 792.3}{s^9 + 6.134s^8 + 441.3s^7 + 2759s^6 + 7.317e^4s^5 + 4.652e^5s^4 + 5.4e^6s^3 + 3.485e^7s^2 1 + .496e^8s + 9.781e^8}$
GA	$\frac{-0.1269s^8 + 2.304s^7 - 102.8s^6 + 1160s^5 - 2.733e^4s^4 + 1.815e^5s^3 - 2.9e^6s^2 + 8.908e^6s - 1.059e^8}{s^9 + 0.4469s^8 + 660.1s^7 + 249.8s^6 + 1.566e^5s^5 + 4.861e^4s^4 + 1.585e^7s^3 + 3.878e^6s^2 + 5.8e^8s + 1.057e^8}$
VS	$\frac{-0.1168s^8 + 2.105s^7 - 97.54s^6 + 1072s^5 - 2.647e^4s^4 + 1.69e^5s^3 - 2.858e^6s^2 + 8.342e^6s - 1.06e^8}{s^9 + 0.4741s^8 + 655s^7 + 271.9s^6 + 1.543e^5s^5 + 5.502e^4s^4 + 1.552e^7s^3 + 4.659e^6s^2 + 5.645e^8s + 1.393e^8}$
PT	$\frac{0.0002851s^8 - 0.002648s^7 + 0.0422s^6 - 0.3299s^5 + 2.665s^4 - 15.2s^3 + 75.93s^2 - 244.1s + 792.3}{s^9 + 9.073s^8 + 463.7s^7 + 3739s^6 + 8.006e^4s^5 + 5.738e^5s^4 + 6.107e^6s^3 + 3.884e^7s^2 + 1.737e^8s + 9.781e^8}$

Table 4.11: 10^{th} Order LOAS

Techniques	LOAS
GJ	$\frac{-8.075e^{-6}s^9 + 0.001304s^8 - 0.000966s^7 + 0.1164s^6 - 0.05621s^5 + 6.153s^4 - 1.551s^3 + 157.7s^2 - 16.55s + 1581}{s^{10} + 0.211s^9 + 593.7s^8 + 104.7s^7 + 1.414e^5s^6 + 1.942e^4s^5 + 1.686e^7s^4 + 1.592e^6s^3 + 1.006e^9s^2 + 4.868e^7s + 2.399e^{10}}$
GA	$\frac{-0.06721s^9 - 0.9587s^8 - 31.95s^7 - 781.4s^6 - 4158s^5 - 2.1e^5s^4 - 6.85e^4s^3 - 2.259e^7s^2 + 9.465e^6s - 8.413e^8}{s^{10} + 0.5127s^9 + 848.9s^8 + 344.7s^7 + 2.784e^5s^6 + 8.336e^4s^5 + 4.409e^7s^4 + 8.606e^6s^3 + 3.378e^9s^2 + 3.21e^8s + 1.004e^{11}}$
VS	$\frac{-0.07711s^9 - 0.6303s^8 - 40.03s^7 - 573.9s^6 - 6419s^5 - 1.634e^5s^4 - 3.279e^5s^3 - 1.817e^7s^2 - 9.834e05s - 6.905e^8}{s^{10} + 0.4907s^9 + 844.2s^8 + 327.6s^7 + 2.754e^5s^6 + 7.869e^4s^5 + 4.34e^7s^4 + 8.076e^6s^3 + 3.312e09s^2 + 2.998e^8s + 9.81e^{10}}$
PT	$\frac{-8.075e^{-6}s^9 + 0.001304s^8 - 0.000966s^7 + 0.1164s^6 - 0.05621s^5 + 6.153s^4 - 1.551s^3 + 157.7s^2 - 16.55s + 1581}{s^{10} + 4.853s^9 + 605.5s^8 + 2264s^7 + 1.453e^5s^6 + 3.945e^5s^5 + 1.729e^7s^4 + 3.043e^7s^3 + 1.022e^9s^2 + 8.769e^8s + 2.399e^{10}}$

Table 4.12: 11^{th} Order LOAS

Techniques	LOAS
GJ	$\frac{-6.631e^{-5}s^{10}+0.001368s^9-0.006677s^8+0.1269s^7 -0.5728s^6+6.891s^5-22.08s^4+181.4s^3-437.2s^2 +1871s-3472}{s^{11}+4.059s^{10}+597.1s^9+2489s^8+1.43e^5s^7 +6.093e^5s^6+1.714e^7s^5+7.437e^7s^4+1.028e^9s^3 +4.524e^9s^2+2.464e^{10}s+1.096e^{11}}$
GA	$\frac{0.08348s^{10}-2.25s^9+94.26s^8-1602s^7+3.575e^4s^6 -3.954e^5s^5+5.903e^6s^4-4.034e^7s^3 +4.349e^8s^2-1.453e^9s+1.162e^{10}}{s^{11}+0.677s^{10}+859.3s^9+483.3s^8+2.85e^5s^7 +1.276e^5s^6+4.562e^7s^5+1.528e^7s^4+3.529e^9s^3 +8.009e^8s^2+1.058e^{11}s+1.32e^{10}}$
VS	$\frac{0.0779s^{10}-2.142s^9+88.87s^8-1529s^7+3.383e^4s^6 -3.775e^5s^5+5.584e^6s^4-3.847e^7s^3 +4.1e^8s^2-1.383e^9s+1.088e^{10}}{s^{11}+0.6917s^{10}+858.4s^9+498.8s^8 +2.845e^5s^7+1.338e^5s^6+4.552e^7s^5+1.646e^7s^4 +3.52e^9s^3+9.061e^8s^2+1.055e^{11}s+1.675e^{10}}$
PT	$\frac{-6.631e^{-5}s^{10}+0.001368s^9-0.006677s^8+0.1269s^7 -0.5728s^6+6.891s^5-22.08s^4+181.4s^3 -437.2s^2+1871s-3472}{s^{11}+9.098s^{10}+630.3s^9+4887s^8+1.57e^5s^7 +1.035e^6s^6+1.935e^7s^5+1.077e^8s^4+1.182e^9s^3 +5.499e^9s^2+2.865e^{10}s+1.096e^{11}}$

Table 4.13: 12^{th} Order LOAS

Techniques	LOAS
GJ	$\frac{7.905e^{-6}s^{11} - 0.0007995s^{10} + 0.0007917s^{9} + 0.009321s^{8} + 0.07658s^{7} - 4.222s^{6} + 3.242s^{5} - 167.6s^{4} + 71.41s^{3} - 3522s^{2} + 641.2s - 3.054e^{4}}{s^{12} + 0.1515s^{11} + 814.4s^{10} + 123.3s^{9} + 2.759e^{5}s^{8} + 3.823e^{4}s^{7} + 4.966e^{7}s^{6} + 5.711e^{6}s^{5} + 5.004e^{9}s^{4} + 4.142e^{8}s^{3} + 2.672e^{11}s^{2} + 1.172e^{10}s + 5.904e^{12}}$
GA	$\frac{0.05336s^{11} - 0.3066s^{10} + 42.72s^{9} - 103.6s^{8} + 1.228e^{4}s^{7} + 1.207e^{4}s^{6} + 1.582e^{6}s^{5} + 7.547e^{6}s^{4} + 9.185e^{7}s^{3} + 8.771e^{8}s^{2} + 1.907e^{9}s + 3.171e^{10}}{s^{12} + 0.7123s^{11} + 1036s^{10} + 608.6s^{9} + 4.343e^{5}s^{8} + 2.009e^{5}s^{7} + 9.425e^{7}s^{6} + 3.203e^{7}s^{5} + 1.118e^{10}s^{4} + 2.472e^{9}s^{3} + 6.879e^{11}s^{2} + 7.404e^{10}s + 1.719e^{13}}$
VS	$\frac{0.05246s^{11} - 0.3728s^{10} + 42.47s^{9} - 162.3s^{8} + 1.237e^{4}s^{7} - 7664s^{6} + 1.623e^{6}s^{5} + 4.392e^{6}s^{4} + 9.702e^{7}s^{3} + 6.361e^{8}s^{2} + 2.115e^{9}s + 2.464e^{10}}{s^{12} + 0.7086s^{11} + 1035s^{10} + 604.8s^{9} + 4.335e^{5}s^{8} + 1.994e^{5}s^{7} + 9.402e^{7}s^{6} + 3.178e^{7}s^{5} + 1.114e^{10}s^{4} + 2.451e^{9}s^{3} + 6.854e^{11}s^{2} + 7.339e^{10}s + 1.712e^{13}}$
PT	$\frac{7.905e^{-6}s^{11} - 0.0007995s^{10} + 0.0007917s^{9} + 0.009321s^{8} + 0.07658s^{7} - 4.222s^{6} + 3.242s^{5} - 167.6s^{4} + 71.41s^{3} - 3522s^{2} + 641.2s - 3.054e^{4}}{s^{12} + 6.2s^{11} + 833.6s^{10} + 4062s^{9} + 2.852e^{5}s^{8} + 1.055e^{6}s^{7} + 5.135e^{7}s^{6} + 1.356e^{8}s^{5} + 5.139e^{9}s^{4} + 8.644e^{9}s^{3} + 2.712e^{11}s^{2} + 2.184e^{11}s + 5.904e^{12}}$

Table 4.14: 13^{th} Order LOAS

Techniques	LOAS
GJ	$\dfrac{5.783e^{-5}s^{12} - 0.001467s^{11} + 0.006104s^{10} -0.05755s^{9} + 1.024s^{8} - 10.39s^{7} + 37.53s^{6} - 421.2s^{5} +1067s^{4} - 8947s^{3} + 1.567e^{4}s^{2} - 7.786e^{4}s +8.889e^{4}}{s^{13} + 2.8s^{12} + 860.6s^{11} + 2544s^{10} + 3.047e^{5}s^{9} +9.374e^{5}s^{8} + 5.686e^{7}s^{7} + 1.799e^{8}s^{6} + 5.898e^{9}s^{5} +1.903e^{10}s^{4} + 3.227e^{11}s^{3} + 1.055e^{12}s^{2} + 7.274e^{12}s +2.395e^{13}}$
GA	$\dfrac{-0.01196s^{12} + 0.5465s^{11} - 27.61s^{10} + 607.3s^{9} -1.73e^{4}s^{8} + 2.371e^{5}s^{7} - 4.653e^{6}s^{6} + 4.146e^{7}s^{5} -6.072e^{8}s^{4} + 3.327e^{9}s^{3} - 3.789e^{10}s^{2} + 9.997e^{10}s -9.043e^{11}}{s^{13} + 0.7879s^{12} + 1047s^{11} + 691.8s^{10} + 4.43e^{5}s^{9} +2.364e^{5}s^{8} + 9.704e^{7}s^{7} + 3.96e^{7}s^{6} + 1.161e^{10}s^{5} +3.324e^{9}s^{4} + 7.201e^{11}s^{3} + 1.222e^{11}s^{2} + 1.812e^{13}s +1.072e^{12}}$
VS	$\dfrac{-0.01096s^{12} + 0.4963s^{11} - 25.32s^{10} + 555.7s^{9} -1.583e^{4}s^{8} + 2.172e^{5}s^{7} - 4.236e^{6}s^{6} + 3.787e^{7}s^{5} -5.48e^{8}s^{4} + 3.021e^{9}s^{3} - 3.38e^{10}s^{2} + 9.012e^{10}s -7.954e^{11}}{s^{13} + 0.7898s^{12} + 1047s^{11} + 695.2s^{10} + 4.435e^{5}s^{9} +2.384e^{5}s^{8} + 9.724e^{7}s^{7} + 4.016e^{7}s^{6} + 1.164e^{10}s^{5} +3.399e^{9}s^{4} + 7.231e^{11}s^{3} + 1.272e^{11}s^{2} + 1.822e^{13}s +1.2e^{12}}$
PT	$\dfrac{5.783e^{-5}s^{12} - 0.001467s^{11} + 0.006104s^{10} - 0.05755s^{9} +1.024s^{8} - 10.39s^{7} + 37.53s^{6} - 421.2s^{5} + 1067s^{4} -8947s^{3} + 1.567e^{4}s^{2} - 7.786e^{4}s +8.889e^{4}}{s^{13} + 6.577s^{12} + 878.3s^{11} + 5148s^{10} + 3.161e^{5}s^{9} +1.644e^{6}s^{8} + 5.97e^{7}s^{7} + 2.743e^{8}s^{6} + 6.25e^{9}s^{5} +2.525e^{10}s^{4} + 3.44e^{11}s^{3} + 1.216e^{12}s^{2} + 7.788e^{12}s +2.395e^{13}}$

Table 4.15: 14^{th} Order LOAS

Techniques	LOAS
GJ	$\frac{-6.984e^{-6}s^{13}+0.0007537s^{12}-0.0007541s^{11}-0.02277s^{10}-0.1416s^{9}+34.6s^{8}-5.668s^{7}+324.2s^{6}-185.3s^{5}+9840s^{4}-3295s^{3}+1.637e^{5}s^{2}-2.458e^{4}s+1.149e^{6}}{s^{14}+0.4486s^{13}+1175s^{12}+465.4s^{11}+5.786e^{5}s^{10}+1.958e^{5}s^{9}+1.55e^{8}s^{8}+4.276e^{7}s^{7}+2.441e^{10}s^{6}+5.113e^{9}s^{5}+2.26e^{12}s^{4}+3.18e^{11}s^{3}+1.141e^{14}s^{2}+8.049e^{12}s+2.423e^{15}}$
GA	$\frac{-0.01089s^{13}+0.3018s^{12}-14.11s^{11}+283.4s^{10}-6730s^{9}+1.021e^{5}s^{8}-1.546e^{6}s^{7}+1.821e^{7}s^{6}-1.852e^{8}s^{5}+1.718e^{9}s^{4}-1.117e^{10}s^{3}+8.166e^{10}s^{2}-2.684e^{11}s+1.526e^{12}}{s^{14}+0.7902s^{13}+1230s^{12}+825.7s^{11}+6.331e^{5}s^{10}+3.496e^{5}s^{9}+1.766e^{8}s^{8}+7.677e^{7}s^{7}+2.885e^{10}s^{6}+9.23e^{9}s^{5}+2.762e^{12}s^{4}+5.767e^{11}s^{3}+1.436e^{14}s^{2}+1.465e^{13}s+3.131e^{15}}$
VS	$\frac{-0.01015s^{13}+0.2855s^{12}-13.19s^{11}+267.3s^{10}-6291s^{9}+9.581e^{4}s^{8}-1.44e^{6}s^{7}+1.697e^{7}s^{6}-1.716e^{8}s^{5}+1.587e^{9}s^{4}-1.028e^{10}s^{3}+7.465e^{10}s^{2}-2.452e^{11}s+1.378e^{12}}{s^{14}+0.7899s^{13}+1230s^{12}+825.6s^{11}+6.333e^{5}s^{10}+3.496e^{5}s^{9}+1.767e^{8}s^{8}+7.679e^{7}s^{7}+2.887e^{10}s^{6}+9.234e^{9}s^{5}+2.764e^{12}s^{4}+5.771e^{11}s^{3}+1.437e^{14}s^{2}+1.466e^{13}s+3.135e^{15}}$
PT	$\frac{-6.984e^{-6}s^{13}+0.0007537s^{12}-0.0007541s^{11}-0.02277s^{10}-0.1416s^{9}+34.6s^{8}-5.668s^{7}+324.2s^{6}-185.3s^{5}+9840s^{4}-3295s^{3}+1.637e^{5}s^{2}-2.458e^{4}s+1.149e^{6}}{s^{14}+0.5456s^{13}+1175s^{12}+555.2s^{11}+5.787e^{5}s^{10}+2.295e^{5}s^{9}+1.55e^{8}s^{8}+4.937e^{7}s^{7}+2.441e^{10}s^{6}+5.828e^{9}s^{5}+2.26e^{12}s^{4}+3.585e^{11}s^{3}+1.141e^{14}s^{2}+8.989e1^{2}s+2.423e^{15}}$

Table 4.16: 15^{th} Order LOAS

Techniques	LOAS
GJ	$\dfrac{\begin{array}{r}-2.865e^{-5}s^{14} + 0.0008359s^{13} - 0.004622s^{12} - 0.01176s^{11} \\ -0.9393s^{10} + 36.46s^{9} + 0.8053s^{8} + 409.6s^{7} \\ -2214s^{6} + 1.277e^{4}s^{5} - 5.959e^{4}s^{4} + 2.182e^{5}s^{3} \\ -8.915e^{5}s^{2} + 1.572e^{6}s - 5.676e^{6}\end{array}}{\begin{array}{r}s^{15} + 2.021s^{14} + 1180s^{13} + 2359s^{12} \\ +5.834e^{5}s^{11} + 1.151e^{6}s^{10} + 1.569e^{8}s^{9} + 3.041e^{8}s^{8} \\ +2.478e^{10}s^{7} + 4.704e^{10}s^{6} + 2.301e^{12}s^{5} + 4.266e^{12}s^{4} \\ +1.164e^{14}s^{3} + 2.102e^{14}s^{2} + 2.48e^{15}s + 4.349e^{15}\end{array}}$
GA	$\dfrac{\begin{array}{r}-0.01209s^{14} + 0.3142s^{13} - 15.61s^{12} + 296.2s^{11} \\ -7511s^{10} + 1.075e^{5}s^{9} - 1.763e^{6}s^{8} + 1.937e^{7}s^{7} \\ -2.201e^{8}s^{6} + 1.854e^{9}s^{5} - 1.443e^{10}s^{4} + 8.997e^{10}s^{3} \\ -4.315e^{11}s^{2} + 1.731e^{12}s - 3.392e^{12}\end{array}}{\begin{array}{r}s^{15} + 0.7919s^{14} + 1230s^{13} + 827.8s^{12} \\ +6.332e^{5}s^{11} + 3.507e^{5}s^{10} + 1.767e^{8}s^{9} + 7.708e^{7}s^{8} \\ +2.886e^{10}s^{7} + 9.279e^{9}s^{6} + 2.763e^{12}s^{5} + 5.813e^{11}s^{4} \\ +1.437e^{14}s^{3} + 1.489e^{13}s^{2} + 3.133e^{15}s + 4.931e^{12}\end{array}}$
VS	$\dfrac{\begin{array}{r}-0.01144s^{14} + 0.3022s^{13} - 14.81s^{12} + 284.4s^{11} \\ -7124s^{10} + 1.028e^{5}s^{9} - 1.669e^{6}s^{8} + 1.846e^{7}s^{7} \\ -2.076e^{8}s^{6} + 1.76e^{9}s^{5} - 1.355e^{10}s^{4} + 8.498e^{10}s^{3} \\ -4.042e^{11}s^{2} + 1.628e^{12}s - 3.193e^{12}\end{array}}{\begin{array}{r}s^{15} + 0.7924s^{14} + 1231s^{13} + 828.7s^{12} \\ +6.335e^{5}s^{11} + 3.512e^{5}s^{10} + 1.768e^{8}s^{9} + 7.725e^{7}s^{8} \\ +2.889e^{10}s^{7} + 9.308e^{9}s^{6} + 2.766e^{12}s^{5} + 5.84e^{11}s^{4} \\ +1.439e^{14}s^{3} + 1.501e^{13}s^{2} + 3.138e^{15}s + 7.231e^{12}\end{array}}$
PT	$\dfrac{\begin{array}{r}-2.865e^{-5}s^{14} + 0.0008359s^{13} - 0.004622s^{12} - 0.01176s^{11} \\ -0.9393s^{10} + 36.46s^{9} + 0.8053s^{8} + 409.6s^{7} \\ -2214s^{6} + 1.277e^{4}s^{5} - 5.959e^{4}s^{4} + 2.182e^{5}s^{3} \\ -8.915e^{5}s^{2} + 1.572e^{6}s - 5.676e^{6}\end{array}}{\begin{array}{r}s^{15} + 2.624s^{14} + 1181s^{13} + 2954s^{12} \\ +5.848e^{5}s^{11} + 1.388e^{6}s^{10} + 1.574e^{8}s^{9} + 3.533e^{8}s^{8} \\ +2.488e^{10}s^{7} + 5.263e^{10}s^{6} + 2.312e^{12}s^{5} + 4.596e^{12}s^{4} \\ +1.171e^{14}s^{3} + 2.182e^{14}s^{2} + 2.494e^{15}s + 4.349e^{15}\end{array}}$

Table 4.17: Poles Locations of LOAS from (1^{st}-15^{th}) order LOAS

LOAS	GJ
1^{st} Order	-0.13866
2^{nd} Order	$-0.0045646 \pm 10.007i$
3^{3rd} Order	$-1.0682, -0.0037271 \pm 10.01i$
4^{th} Order	$0.04473 \pm 10.086i, -0.055275 \pm 10.062i$
5^{th} Order	$-2.4349, 0.041504 \pm 10.116i, -0.036295 \pm 10.062i$
6^{th} Order	$0.27434 \pm 10.316i, -0.28979 \pm 10.282i, -0.010912 \pm 10.048i$
7^{th} Order	$-4.0193, -0.010747 \pm 10.048i, -0.24253 \pm 10.319i, 0.304 \pm 10.375i$
8^{th} Order	$0.65462 \pm 10.75i, -0.67391 \pm 10.695i, -0.02581 \pm 10.404i, -0.011455 \pm 10.049i$
9^{th} Order	$-6.4415, 0.73491 \pm 10.885i, -0.5536 \pm 10.77i, -0.01601 \pm 10.422i, -0.011461 \pm 10.049i$
10^{th} Order	$1.1605 \pm 11.536i, -1.1892 \pm 11.456i, -0.03056 \pm 11.043i, -0.034795 \pm 10.451i, -0.011435 \pm 10.049i$
11^{th} Order	$-4.4126, -0.011435 \pm 10.049i, -0.035273 \pm 10.45i, 0.008128 \pm 11.082i, -1.0363 \pm 11.492i, 1.2516 \pm 11.671i$
12^{th} Order	$1.5122 \pm 13.057i, -1.4712 \pm 12.869i, -0.012079 \pm 12.086i, -0.059097 \pm 11.247i, -0.034161 \pm 10.448i, -0.011435 \pm 10.049i$
13^{th} Order	$-3.2806, -0.011435 \pm 10.049i, -0.034205 \pm 10.448i, -0.047599 \pm 11.23i, -0.61062 \pm 12.341i, 0.61017 \pm 12.646i, 0.33423 \pm 14.644i$
14^{th} Order	$-0.013441 \pm 16.168i, 0.024267 \pm 15.821i, -0.018532 \pm 12.975i, -0.11566 \pm 12.573i, -0.055314 \pm 11.235i, -0.034173 \pm 10.448i, -0.011435 \pm 10.049i$
15^{th} Order	$-1.761, -0.011435 \pm 10.049i, -0.034175 \pm 10.448i, -0.054623 \pm 11.236i, -0.16944 \pm 12.502i, 0.10693 \pm 13.063i, 0.043786 \pm 15.849i, -0.010985 \pm 16.27i$

Table 4.18: Poles Locations of LOAS from (1^{st}-15^{th}) order LOAS

LOAS	GA
1^{st} Order	-0.013708
2^{nd} Order	$-0.010148 \pm 10.121i$
3^{3rd} Order	$-1.0682, -0.0037271 \pm 10.01i$
4^{th} Order	$-0.016836 \pm 11.286i, -0.01168 \pm 10.116i$
5^{th} Order	$-0.015313, -0.011534 \pm 10.118i, -0.016224 \pm 11.319i$
6^{th} Order	$-0.013257 \pm 14.066i, -0.019578 \pm 11.284i - 0.012243 \pm 10.114i$
7^{th} Order	$-0.14293, -0.011547 \pm 10.125i, -0.015477 \pm 11.411i, -0.0096191 \pm 14.338i$
8^{th} Order	$-0.027058 \pm 15.737i, -0.039528 \pm 13.257i, -0.034455 \pm 11.082i, -0.014929 \pm 10.096i$
9^{th} Order	$-0.18238, -0.01585 \pm 10.1i, -0.039582 \pm 11.149i, -0.046984 \pm 13.446i, -0.02987 \pm 15.904i$
10^{th} Order	$-0.038303 \pm 16.415i, -0.074675 \pm 14.551i, -0.077022 \pm 12.321i, -0.051626 \pm 10.705i, -0.014706 \pm 10.059i$
11^{th} Order	$-0.12485, -0.014912 \pm 10.059i, -0.055536 \pm 10.723i, -0.085944 \pm 12.418i, -0.081589 \pm 14.71i, -0.038118 \pm 16.502i$
12^{th} Order	$-0.035035 \pm 16.716i, -0.087104 \pm 15.346i, -0.10256 \pm 13.326i, -0.079696 \pm 11.528i, -0.040449 \pm 10.469i, -0.011298 \pm 10.049i$
13^{th} Order	$-0.059171, -0.011209 \pm 10.05i, -0.041875 \pm 10.472i, -0.085566 \pm 11.584i, -0.10744 \pm 13.466i, -0.085939 \pm 15.476i, -0.032329 \pm 16.754i$
14^{th} Order	$-0.026293 \pm 16.856i, -0.077306 \pm 15.883i, -0.1024 \pm 14.198i, -0.087303 \pm 12.473i, -0.056118 \pm 11.236i, -0.034213 \pm 10.451i, -0.011486 \pm 10.049i$
15^{th} Order	$-0.0015737, -0.011481 \pm 10.049i, -0.034263 \pm 10.451i, -0.056184 \pm 11.238i, -0.087137 \pm 12.475i, -0.10243 \pm 14.198i, -0.077376 \pm 15.884i, -0.026272 \pm 16.857i$

Table 4.19: Poles Locations of LOAS from (1^{st}-15^{th}) order LOAS

LOAS	GS
1^{st} Order	-0.0050659
2^{nd} Order	$-0.0090048 \pm 10.104i$
3^{3rd} Order	$-0.024621, -0.008817 \pm 10.106i$
4^{th} Order	$-0.016788 \pm 11.144i, -0.01015 \pm 10.102i$
5^{th} Order	$-0.026425, -0.01548 \pm 11.165i, -0.0098249 \pm 10.104i$
6^{th} Order	$-0.012261 \pm 13.805i, -0.018019 \pm 11.148i, -0.010443 \pm 10.102i$
7^{th} Order	$-0.0069508, -0.010471 \pm 10.102i, -0.018214 \pm 11.154i, -0.012807 \pm 13.838i$
8^{th} Order	$-0.026545 \pm 15.632i, -0.036331 \pm 13.095i, -0.029898 \pm 11.001i, -0.012576 \pm 10.091i$
9^{th} Order	$-0.18238, -0.01585 \pm 10.1i, -0.039582 \pm 11.149i, -0.046984 \pm 13.446i, -0.02987 \pm 15.904i$
10^{th} Order	$-0.038303 \pm 16.415i, -0.074675 \pm 14.551i, -0.077022 \pm 12.321i, -0.051626 \pm 10.705i, -0.014706 \pm 10.059i$
11^{th} Order	$-0.12485, -0.014912 \pm 10.059i, -0.055536 \pm 10.723i, -0.085944 \pm 12.418i, -0.081589 \pm 14.71i, -0.038118 \pm 16.502i$
12^{th} Order	$-0.035035 \pm 16.716i, -0.087104 \pm 15.346i, -0.10256 \pm 13.326i, -0.079696 \pm 11.528i, -0.040449 \pm 10.469i, -0.011298 \pm 10.049i$
13^{th} Order	$-0.059171, -0.011209 \pm 10.05i, -0.041875 \pm 10.472i, -0.085566 \pm 11.584i, -0.10744 \pm 13.466i, -0.085939 \pm 15.476i, -0.032329 \pm 16.754i$
14^{th} Order	$-0.026293 \pm 16.856i, -0.077306 \pm 15.883i, -0.1024 \pm 14.198i, -0.087303 \pm 12.473i, -0.056118 \pm 11.236i, -0.034213 \pm 10.451i, -0.011486 \pm 10.049i$
15^{th} Order	$-0.0015737, -0.011481 \pm 10.049i, -0.034263 \pm 10.451i, -0.056184 \pm 11.238i, -0.087137 \pm 12.475i, -0.10243 \pm 14.198i, -0.077376 \pm 15.884i, -0.026272 \pm 16.857i$

Table 4.20: Poles Locations of LOAS from (1^{st}-15^{th}) order LOAS

LOAS	Proposed Technique
1^{st} Order	-0.13866
2^{nd} Order	$-0.0045646 \pm 10.007i$
3^{3rd} Order	$-1.0682, -0.0037271 \pm 10.01i$
4^{th} Order	$0.04473 \pm 10.086i, -0.055275 \pm 10.062i$
5^{th} Order	$-2.4349, -0.036295 \pm 10.062i, -0.041504 \pm 10.116i$
6^{th} Order	$-0.27434 \pm 10.316i, -0.28979 \pm 10.282i, -0.010912 \pm 10.048i$
7^{th} Order	$-4.0193, -0.010747 \pm 10.048i, -0.24253 \pm 10.319i, 0.304 \pm 10.375i$
8^{th} Order	$-0.65462 \pm 10.75i, -0.67391 \pm 10.695i, -0.02581 \pm 10.404i, -0.011455 \pm 10.049i$
9^{th} Order	$-6.4415, -0.73491 \pm 10.885i, -0.5536 \pm 10.77i, -0.01601 \pm 10.422i, -0.011461 \pm 10.049i$
10^{th} Order	$-1.1605 \pm 11.536i, -1.1892 \pm 11.456i, -0.03056 \pm 11.043i, -0.034795 \pm 10.451i, -0.011435 \pm 10.049i$
11^{th} Order	$-4.4126, -0.011435 \pm 10.049i, -0.035273 \pm 10.45i, -0.008128 \pm 11.082i, -1.0363 \pm 11.492i, -1.2516 \pm 11.671i$
12^{th} Order	$-1.5122 \pm 13.057i, -1.4712 \pm 12.869i, -0.012079 \pm 12.086i, -0.059097 \pm 11.247i, -0.034161 \pm 10.448i, -0.011435 \pm 10.049i$
13^{th} Order	$-3.2806, -0.011435 \pm 10.049i, -0.034205 \pm 10.448i, -0.047599 \pm 11.23i, -0.61062 \pm 12.341i, -0.61017 \pm 12.646i, -0.33423 \pm 14.644i$
14^{th} Order	$-0.013441 \pm 16.168i, -0.024267 \pm 15.821i, -0.018532 \pm 12.975i, -0.11566 \pm 12.573i, -0.055314 \pm 11.235i, -0.034173 \pm 10.448i, -0.011435 \pm 10.049i$
15^{th} Order	$-1.761, -0.011435 \pm 10.049i, -0.034175 \pm 10.448i, -0.054623 \pm 11.236i, -0.16944 \pm 12.502i, -0.10693 \pm 13.063i, -0.043786 \pm 15.849i, -0.010985 \pm 16.27i$

Table 4.21: Poles Locations of LOAS from (1^{st}-15^{th}) order LOAS

LOAS	GJ	GA	GS	Proposed Technique
1^{st} Order	-0.13866	-0.013708	-0.0050659	-0.13866
2^{nd} Order	$-0.0045646 \pm 10.007i$	$-0.010148 \pm 10.121i$	$-0.0090048 \pm 10.104i$	$-0.0045646 \pm 10.007i$
3^{3rd} Order	$-1.0682, -0.0037271 \pm 10.01i$	$-0.008555, -0.010119 \pm 10.122i$	$-0.024621, -0.008817 \pm 10.106i$	$-1.0682, -0.0037271 \pm 10.01i$
4^{th} Order	$0.04473 \pm 10.086i, -0.055275 \pm 10.062i$	$-0.016836 \pm 11.286i, -0.01168 \pm 10.116i$	$-0.016788 \pm 11.144i, -0.01015 \pm 10.102i$	$0.04473 \pm 10.086i, -0.055275 \pm 10.062i$
5^{th} Order	$-2.4349, 0.041504 \pm 10.116i, -0.036295 \pm 10.062i$	$-0.015313, -0.011534 \pm 10.118i, -0.016224 \pm 11.319i$	$-0.026425, -0.01548 \pm 11.165i, -0.0098249 \pm 10.104i$	$-2.4349, -0.036295 \pm 10.062i, -0.041504 \pm 10.116i$
6^{th} Order	$0.27434 \pm 10.316i, -0.28979 \pm 10.282i, -0.010912 \pm 10.048i$	$-0.013257 \pm 14.066i, -0.019578 \pm 11.284i - 0.012243 \pm 10.114i$	$-0.012261 \pm 13.805i, -0.018019 \pm 11.148i, -0.010443 \pm 10.102i$	$-0.27434 \pm 10.316i, -0.28979 \pm 10.282i, -0.010912 \pm 10.048i$
7^{th} Order	$-4.0193, -0.010747 \pm 10.048i, -0.24253 \pm 10.319i, 0.304 \pm 10.375i$	$-0.14293, -0.011547 \pm 10.125i, -0.015477 \pm 11.411i, -0.0096191 \pm 14.338i$	$-0.0069508, -0.010471 \pm 10.102i, -0.018214 \pm 11.154i, -0.012807 \pm 13.838i$	$-4.0193, -0.010747 \pm 10.048i, -0.24253 \pm 10.319i, 0.304 \pm 10.375i$
8^{th} Order	$0.65462 \pm 10.75i, -0.67391 \pm 10.695i, -0.02581 \pm 10.404i, -0.011455 \pm 10.049i$	$-0.027058 \pm 15.737i, -0.039528 \pm 13.257i, -0.034455 \pm 11.082i, -0.014929 \pm 10.096i$	$-0.026545 \pm 15.632i, -0.036331 \pm 13.095i, -0.029898 \pm 11.001i, -0.012576 \pm 10.091i$	$-0.65462 \pm 10.75i, -0.67391 \pm 10.695i, -0.02581 \pm 10.404i, -0.011455 \pm 10.049i$
9^{th} Order	$-6.4415, 0.73491 \pm 10.885i, -0.5536 \pm 10.77i, -0.01601 \pm 10.422i, -0.011461 \pm 10.049i$	$-0.18238, -0.01585 \pm 10.1i, -0.039582 \pm 11.149i, -0.046984 \pm 13.446i, -0.02987 \pm 15.904i$	$-0.24689, -0.013492 \pm 10.1i, -0.033564 \pm 11.113i, -0.040445 \pm 13.364i, -0.026127 \pm 15.837i$	$-6.4415, -0.73491 \pm 10.885i, -0.5536 \pm 10.77i, -0.01601 \pm 10.422i, -0.011461 \pm 10.049i$
10^{th} Order	$1.1605 \pm 11.536i, -1.1892 \pm 11.456i, -0.03056 \pm 11.043i, -0.034795 \pm 10.451i, -0.011435 \pm 10.049i$	$-0.038303 \pm 16.415i, -0.074675 \pm 14.551i, -0.077022 \pm 12.321i, -0.051626 \pm 10.705i, -0.014706 \pm 10.059i$	$-0.038762 \pm 16.38i, -0.072406 \pm 14.477i, -0.072037 \pm 12.267i, -0.0477 \pm 10.7i, -0.014464 \pm 10.062i$	$-1.1605 \pm 11.536i, -1.1892 \pm 11.456i, -0.03056 \pm 11.043i, -0.034795 \pm 10.451i, -0.011435 \pm 10.049i$
11^{th} Order	$-4.4126, -0.011435 \pm 10.049i, -0.035273 \pm 10.45i, 0.008128 \pm 11.082i, -1.0363 \pm 11.492i, 1.2516 \pm 11.671i$	$-0.12485, -0.014912 \pm 10.059i, -0.055536 \pm 10.723i, -0.085944 \pm 12.418i, -0.081589 \pm 14.71i, -0.038118 \pm 16.502i$	$-0.15883, -0.015056 \pm 10.063i, -0.053233 \pm 10.736i, -0.082327 \pm 12.415i, -0.078604 \pm 14.689i, -0.037203 \pm 16.485i$	$-4.4126, -0.011435 \pm 10.049i, -0.035273 \pm 10.45i, -0.008128 \pm 11.082i, -1.0363 \pm 11.492i, -1.2516 \pm 11.671i$
12^{th} Order	$1.5122 \pm 13.057i, -1.4712 \pm 12.869i, -0.012079 \pm 12.086i, -0.059097 \pm 11.247i, -0.034161 \pm 10.448i, -0.011435 \pm 10.049i$	$-0.035035 \pm 16.716i, -0.087104 \pm 15.346i, -0.10256 \pm 13.326i, -0.079696 \pm 11.528i, -0.040449 \pm 10.469i, -0.011298 \pm 10.049i$	$-0.035365 \pm 16.711i, -0.087128 \pm 15.329i, -0.10137 \pm 13.31i, -0.078224 \pm 11.53i, -0.040783 \pm 10.475i, -0.011434 \pm 10.049i$	$-1.5122 \pm 13.057i, -1.4712 \pm 12.869i, -0.012079 \pm 12.086i, -0.059097 \pm 11.247i, -0.034161 \pm 10.448i, -0.011435 \pm 10.049i$
13^{th} Order	$-3.2806, -0.011435 \pm 10.049i, -0.034205 \pm 10.448i, -0.047599 \pm 11.23i, -0.61062 \pm 12.341i, 0.61017 \pm 12.646i, 0.33423 \pm 14.644i$	$-0.059171, -0.011209 \pm 10.05i, -0.041875 \pm 10.472i, -0.085566 \pm 11.584i, -0.10744 \pm 13.466i, -0.085939 \pm 15.476i, -0.032329 \pm 16.754i$	$-0.065879, -0.011368 \pm 10.049i, -0.043121 \pm 10.482i, -0.084678 \pm 11.606i, -0.10558 \pm 13.474i, -0.084868 \pm 15.469i, -0.032339 \pm 16.749i$	$-3.2806, -0.011435 \pm 10.049i, -0.034205 \pm 10.448i, -0.047599 \pm 11.23i, -0.61062 \pm 12.341i, -0.61017 \pm 12.646i, -0.33423 \pm 14.644i$
14^{th} Order	$-0.013441 \pm 16.168i, 0.024267 \pm 15.821i, -0.018532 \pm 12.975i, -0.11566 \pm 12.573i, -0.055314 \pm 11.235i, -0.034173 \pm 10.448i, -0.011435 \pm 10.049i$	$-0.026293 \pm 16.856i, -0.077306 \pm 15.883i, -0.1024 \pm 14.198i, -0.087303 \pm 12.473i, -0.056118 \pm 11.236i, -0.034213 \pm 10.451i, -0.011486 \pm 10.049i$	$-0.026311 \pm 16.856i, -0.077115 \pm 15.883i, -0.10207 \pm 14.202i, -0.087473 \pm 12.479i, -0.056433 \pm 11.236i, -0.034054 \pm 10.451i, -0.011514 \pm 10.049i$	$-0.013441 \pm 16.168i, -0.024267 \pm 15.821i, -0.018532 \pm 12.975i, -0.11566 \pm 12.573i, -0.055314 \pm 11.235i, -0.034173 \pm 10.448i, -0.011435 \pm 10.049i$
15^{th} Order	$-1.761, -0.011435 \pm 10.049i, -0.034175 \pm 10.448i, -0.054623 \pm 11.236i, -0.16944 \pm 12.502i, 0.10693 \pm 13.063i, 0.043786 \pm 15.849i, -0.010985 \pm 16.27i$	$-0.0015737, -0.011481 \pm 10.049i, -0.034263 \pm 10.451i, -0.056184 \pm 11.238i, -0.087137 \pm 12.475i, -0.10243 \pm 14.198i, -0.077376 \pm 15.884i, -0.026272 \pm 16.857i$	$-0.0023045, -0.011507 \pm 10.049i, -0.034148 \pm 10.451i, -0.056492 \pm 11.238i, -0.087233 \pm 12.48i, -0.10215 \pm 14.202i, -0.077232 \pm 15.884i, -0.026268 \pm 16.857i$	$-1.761, -0.011435 \pm 10.049i, -0.034175 \pm 10.448i, -0.054623 \pm 11.236i, -0.16944 \pm 12.502i, -0.10693 \pm 13.063i, -0.043786 \pm 15.849i, -0.010985 \pm 16.27i$

Chapter 5

CONCLUSION AND FUTURE WORK DIRECTIONS

5.1 Conclusion

Improved frequency limited state feedback controller of LOAS technique for discrete time systems is suggested. The proposed technique provides stability of LOAS and gives error bounds. The estimated error is far less as compared to other conventional existing techniques for preserving stability. WZ's technique provide low error for approximation as compared to others, but it occasionally gives unstable LOAS. Improved frequency limited observer based state feedback controller is also presented for discrete time systems of LOAS comparison shows the effectiveness of both procedure as compared to other techniques. Both techniques provide better as results in terms of natural response, impulse response, step response as compared to other techniques which shows the effectiveness of proposed algorithms.

Improved frequency limited state feedback controller of LOAS technique for continuous time systems is suggested. This technique validates stability of LOAS and thus gives also error bounds. The error for approximation is far less as compared to other conventional existing stability preserving techniques as of today. Technique of GJ's provide low error for approximation as compared to other techniques, but again, gives LOAS which are unstable. Improved frequency limited observer based state feedback controller is also presented for discrete time systems of LOAS comparison shows the effectiveness of both procedure as compared to other techniques. Both techniques provide better as results in terms of natural response, impulse response, step response as compared to other techniques which shows the effectiveness of proposed algorithms.

5.2 Future Work and Directions

For future directions, it is recommended:

- To use static state feedback controller of LOAS for nonlinear systems.
- To use dynamic state feedback controller of LOAS for nonlinear systems.
- To use static observer based feedback controller of LOAS for nonlinear systems.
- To use dynamic observer based feedback controller of LOAS for nonlinear systems.
- To use static state feedback controller of LOAS for incase of exact feedback linearization of nonlinear systems.
- To use dynamic state feedback controller of LOAS for incase of exact feedback linearization of nonlinear systems.
- To explore integral static state feedback controller of LOAS for nonlinear systems.
- To build integral dynamic state feedback controller of LOAS for nonlinear systems.
- To find integral static observer based feedback controller of LOAS for nonlinear systems.

- To use integral dynamic observer based feedback controller of LOAS for nonlinear systems.
- To use integral static state feedback controller of LOAS for incase of exact feedback linearization of nonlinear systems.
- To explore integral dynamic state feedback controller of LOAS for incase of exact feedback linearization of nonlinear systems.

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